OPTIMIZING STUDENT ENGAGEMENT WITH DYNAMICS THROUGH SIMULATION SOFTWARE

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Abstract

The wide availability of simulation software, such as MAPLE®, has made it possible to significantly enhance the teaching of courses in mechanics. More challenging problems, usually non-linear, can be introduced which previously could not be treated because of the lack of analytic solutions. By means of numerical solutions to these problems, students can get a feel for finite difference approaches and, perhaps more importantly, their physical understanding can be enhanced and new phenomena explored. The following examples are presented with the underlying equations cast, as much as possible, into non-dimensional form: (i) a finite difference scheme involving a non-linear pendulum subjected to distinct initial conditions, showing that the period depends on the amplitude; (ii) a simple pendulum on an oscillating support, illustrating parametric resonance; and (iii) forced harmonic motion of a non-linear hardening spring-mass system, illustrating the jump phenomena.

Introduction

The use of software as a teaching aid in undergraduate mechanical engineering courses has been discussed by several authors. A common type used is mathematics software which allows for a wide range of applications from basic to advanced engineering courses. MATLAB®[1], MATHCAD® [2] and MAPLE® [3] are some common examples.

In a previous paper[4] the author discussed his experiences on using commercially available simulation software for dynamics teaching and improving learning. This was done using MSC ADAMS®[5] for rigid body dynamics. Feedback from students showed that simulations can help visualization and understanding of mechanical systems dynamics.

There are numerous papers on the use of software in engineering undergraduate courses. For example in reference[6] Gharghouri discusses his experiences on using MAPLE® for teaching a numerical analysis course. An approach to calculate eigenvalues and eigenvectors for a mechanical vibrations course is given and possible advantages of using the software as opposed to programming languages is discussed. It was concluded that the use of the software greatly enhanced the delivery of the course.

Another example of a positive use of MAPLE® as a teaching aid can be found in reference[7] where Gerber discusses how the software aided in teaching circuits and systems.

In this paper a couple of examples developed utilizing MAPLE® are presented. The goal is to improve student understanding of mechanical vibrations and dynamics by investigating non-linear physical phenomena and resonance with the aid of the software.

MAPLE® Examples

Finite difference scheme – simple pendulum

In the following examples solutions are obtained using a MAPLE® finite difference scheme. The algorithm utilized for the numerical integrations is a Fehlberg fourth-fifth order Runge-Kutta.

To give students some sense of a finite difference process, an example problem is solved using Euler's method (see, for example, reference[8]).

Consider the first order differential equation:

$$\frac{dy(t)}{dt} = -K \ y(t) + f(t)$$

(1)

In the Euler's method, the time range of interest is divided into uniform intervals of magnitude h.

Starting at some time t_0, the value of $y(t_0 + h)$ can be approximated by the value of $y(t_0)$ plus the time step h multiplied by the slope of the function, which is the derivative of $y(t)$ (this is simply a first order Taylor expansion):

$$y(t_0 + h) \approx y(t_0) + h \left. \frac{dy(t)}{dt} \right|_{t=t_0}$$

(2)

Then, by calculating the value of $\frac{dy(t)}{dt}$ at time t_0 (using equation (1)), an approximation for the value of y at time $t_0 + h$ can be obtained using equation (2).

The process can then be repeated to generate an approximate solution for the equation. This is shown graphically for a single process iteration in Figure 1.

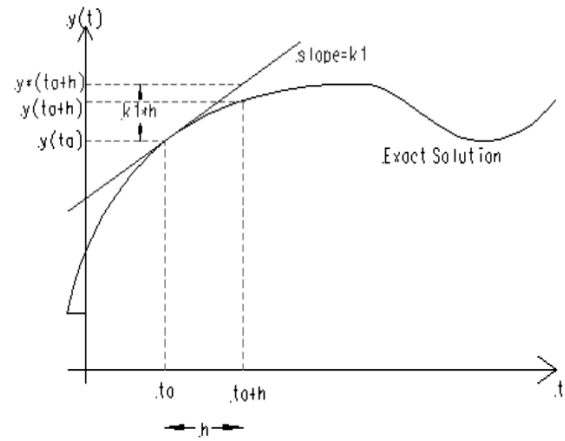Next consider the motion of a simple pendulum (see Figure 2).

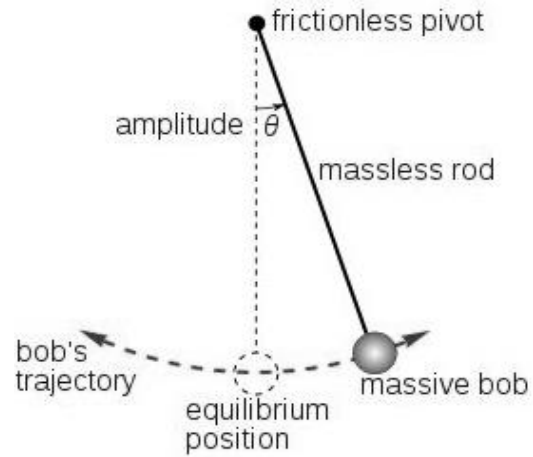Figure 1. Finite difference scheme – Euler's method.

Figure 2. Simple pendulum.

The equation of motion is:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0$$

(3)

where g is the gravitational constant, L is the length of the pendulum and θ is an angular coordinate. Noting that $\sqrt{g/L}$ has dimensions of frequency, one can introduce the dimensionless time: $\tau = (\sqrt{g/L})t$, so that $d/dt = (d/d\tau)(d\tau/dt) = (\sqrt{g/L})(d/d\tau)$.

Then equation (3) becomes:

$$\frac{d^2\theta}{d\tau^2} + \sin(\theta) = 0$$

(4)

A common assumption, utilized when discussing this problem in a classroom, is to assume small oscillations ($\theta \leq 10^{\circ}$, so $\sin(\theta) \approx \theta$). In this case, equation (4) is linear and the dimensionless period is, regardless of initial amplitude, $T_L = 2\pi$. The equation becomes:

$$\frac{d^2\theta}{d\tau^2} + \theta = 0$$

(5)

which has a general solution of the form: $C_1 \sin(\tau) + C_2 \cos(\tau)$. The constant values C_1 and C_2 depend on the initial conditions.

Consider, initially, the following set of initial conditions: $\theta(0) = 4.5^{\circ}$ and $d\theta/d\tau\ (0) = 0$. Then the exact solution for equation (5) is: $0.07853\cos(\tau)$.

Next, a numerical solution for the equation is obtained using Euler's method. Equation (5) is a second order differential equation, therefore, in order to apply the method, it is written as a system of first-order equations:

$$\begin{cases} d\theta_1(\tau)/d\tau = \theta_2(\tau) \\ d\theta_2(\tau)/d\tau = -\theta_1(\tau) \end{cases}$$

(6)

where $\theta_1(\tau) = \theta(\tau)$ and $\theta_2(\tau) = d\theta(\tau)/d\tau$.

The approach leads to:

$$\begin{cases} \theta_1(i+1) = \theta_1(i) + h(\theta_2(i)) \\ \theta_2(i+1) = \theta_2(i) - h(\theta_1(i)) \end{cases}$$

(7)

Taking $h = 0.01$, the numerical solution is plotted together with the exact solution in Figure 3 (see MAPLE® worksheet in the appendix).
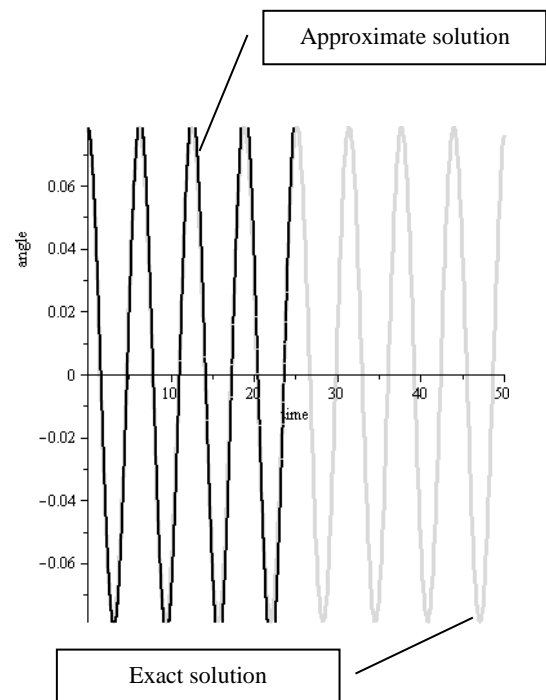

Figure 3. Comparison: Euler's method versus exact solution.

As it can be seen, excellent agreement is obtained.

The example illustrates how a finite difference scheme can be applied to solve a differential equation. Students should realize that numerical schemes can also be used to solve non-linear equations, which may not have closed-form solutions. Thus a deeper insight into the physics of these problems can be obtained by this approach.

For example, consider another set of initial conditions: $\theta(0) = 45°$ and $d\theta/d\tau\,(0) = 0$. In this case the initial angle is not small and the previous assumption that led to a linear equation does not apply anymore. Equation (4) must be used instead. With the aid of MAPLE® the equation can be solved numerically for this and any other set of initial conditions (the default Runge-Kutta solver is used in the following).
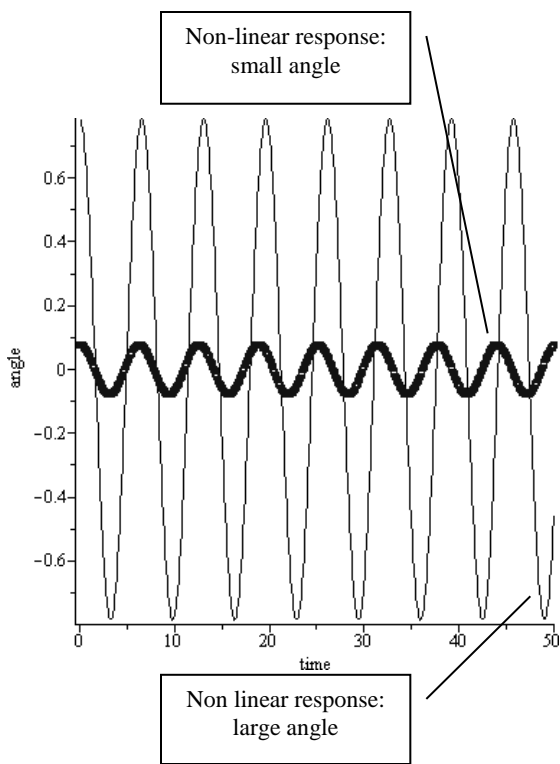

Figure 4. Non-linear response for small and large initial angles.

Figure 4 shows the numerical solution of equation (4) for the two sets of initial conditions mentioned above: small and large initial angles.

It can be seen that the period for "large" response is distinctly different from the "small" one, showing the dependence of the period on the initial condition. Students should note that for a non-linear system the period is not a system characteristic. It depends on the initial amplitude. (Unlike the period of the linear equation, which, independent of the initial conditions, is always 2π).

These results can be explained intuitively as follows. In the range $0 \le \theta \le \pi/2$, θ is greater than $\sin(\theta)$. Hence the effective restoring force in the non-linear case is less than that in the linear case. The motion slows down; the period is larger.

Motion of a pendulum on an oscillating support

Resonances other than forced motion ones can be illustrated with the following example.

Consider the motion of a pendulum which is connected to a support that undergoes a harmonic translational motion. The pendulum is subjected to gravity and to a viscous damping moment at the support ($cl^2\,d\theta/dt$). The amplitude of the translational motion and its frequency are prescribed, ql and Ω, respectively. The system is shown in Figure 5.
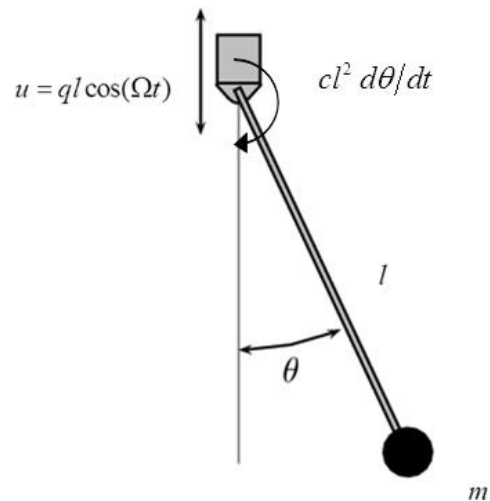

Figure 5. Pendulum on an oscillating support.

Derivation of the equation of motion may be accomplished by using a reference frame translating with u. An observer in that frame sees the mass m undergoing circular motion. Then it can be seen that the acceleration of the end mass is given by:

$$\vec{a} = (\ddot{u}\cos(\theta) + l\dot{\theta}^2)\vec{n} + (\ddot{u}\sin(\theta) + l\ddot{\theta})\vec{t}$$

where $\vec{n}$ and $\vec{t}$ are unit vectors directed along the normal and tangential directions to the circular motion, respectively.

Using $\sum F_t = ma_t$ leads to:

$-mg\sin(\theta) + R_t = m(\ddot{u}\sin(\theta) + l\ddot{\theta})$ where R_t is the pin reaction force along $\vec{t}$. Taking moments about the center of mass (here the end point mass) leads to: $-R_t l - cl^2\dot{\theta} = 0$.

Solving this equation for R_t and substituting into the previous one leads to the equation of motion:

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{1}{l}(g + \ddot{u})\sin(\theta) = 0 \tag{8}$$

After substituting the function for the harmonic translation and assuming small pendulum oscillations ($\sin(\theta) \approx \theta$), a non-dimensional version of equation (8) can be written as:

$$\frac{d^2\theta}{d\tau^2} + C_1\frac{d\theta}{d\tau} + (1 - \upsilon^2 q\cos(\upsilon\tau))\theta = 0 \tag{9}$$

where the following non-dimensional coefficients were employed:

$$\tau = \Omega_0 t \,,\, \Omega_0^2 = g/l \,,\, C_1 = c/\Omega_0 m \,,\, \upsilon = \Omega/\Omega_0$$

Equation (9) is a damped Mathieu-Hill [9] equation. Students should be made aware that, although the system is homogeneous, with no apparent forcing function, there are situations when the response of this type of system can be unstable. This is due to the presence of the time-dependent coefficient in the equation. It can be shown that the coefficient frequencies that will cause instabilities are related to the undamped natural frequency of the system by $\upsilon = 2\upsilon_n/j$, $j = 1,2,3...$ (where υ_n is the non-dimensional undamped natural frequency, see reference [9] for more details). Note that here $\upsilon_n = \sqrt{(g/\Omega_0^2 l)} = 1$, then the condition becomes $\upsilon = 2/j$, $j = 1,2,3...$.

When this is satisfied, a so-called "parametric resonance" is possible. For $j = 1$ the condition leads to the primary parametric resonance at $\upsilon = 2$. This is different from a forced resonance condition, where a resonance is expected when the frequency of the forcing function approaches the natural frequency of the system ($\upsilon = 1$ not $\upsilon = 2$). This parametric instability is verified numerically in the following (MAPLE® worksheet given in the appendix).

Figure 6 shows the response for the following parameters: $\theta(0) = 0.1rad$, $d\theta(0)/d\tau = 0$, $q = 0.2$, $C_1 = 0.01$ and $\upsilon = 2.0$ which, as mentioned above, is expected to lead to unstable response. The numerical simulation shows an exponential growth of the response and, consequently, instability. Figure 7 shows the numerically obtained response for the following parameters: $\theta(0) = 0.1rad$, $d\theta(0)/d\tau = 0$, $q = 0.2$ and $\upsilon = 1.6$. In this case the condition for parametric instability is not satisfied and no resonance is expected. This is confirmed by the response shown in the figure.
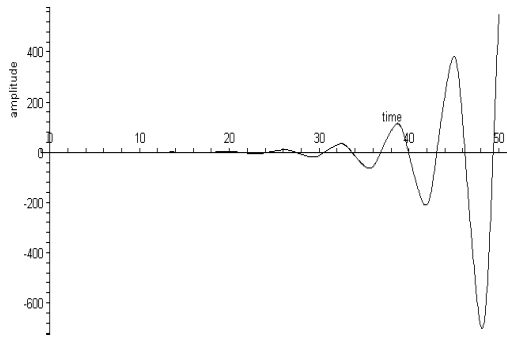
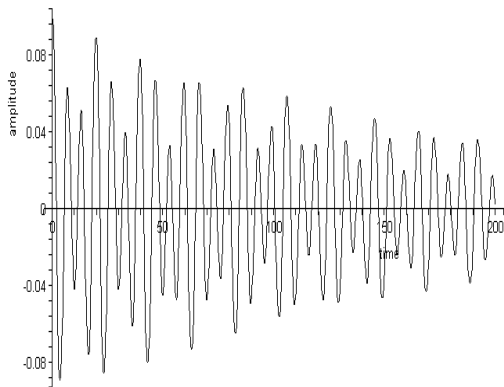Figure 6. Pendulum response to initial conditions – unstable.

Figure 7. Pendulum response to initial conditions – stable.

There are several approaches to investigating the conditions that drive parametric systems unstable. Their discussion is beyond this text. Nevertheless, a simple approach, known as Hill's infinite determinant [9], is shown here. This approach can lead to the "boundaries" of the instability zones in the space defined by the parameters υ versus q. Students can verify, numerically, whether the response of the system to initial conditions is stable or not (and confirm the predictions given by the approach).
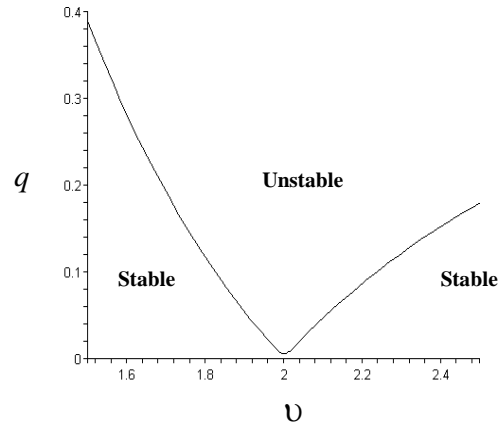

Figure 8. Pendulum stability boundaries.

The primary instability zone ($j = 1$) for the pendulum system is given in Figure 8 (using the same numerical values as before). When the system parameters q and υ lead to a point falling inside the "unstable region" the response to small disturbances will be unstable. Note that the unstable response obtained above is for the point $q = 0.2$ and $\upsilon = 2.0$, which can be seen to fall inside the unstable region. On the other hand, the stable response is for a point falling in the stable region ($q = 0.2$ and $\upsilon = 1.6$).

Damping does play a role here and enough damping could stabilize an "unstable condition" (the instability zone moves off the υ axis).

Harmonic motion of a hardening spring-damper-mass system

An important consequence of non-linearity can be illustrated with the following example. The equation of motion for a spring-damper-mass system with a hardening spring is given by:

$$\frac{d^2x}{dt^2} + 2\beta\omega_0\,\frac{dx}{dt} + \omega_0{}^2x + \frac{k_1}{m}x^3 = \frac{Q}{m}\sin(\omega t)$$

(10)

where the spring force is $kx + k_1 x^3$, m is the mass, β is the damping ratio of the system, ω_0 (given by $\sqrt{(k/m)}$) is the undamped linear natural frequency and Q and ω are the amplitude and frequency, respectively, of the external harmonic excitation. Setting $k_1 = \delta k$ and introducing the following dimensionless quantities: $\tau = \omega_0 t$, $\chi = x/(Q/k)$, $\nu = \omega/\omega_0$; equation (10) becomes:

$$\frac{d^2\chi}{d\tau^2} + 2\beta \frac{d\chi}{d\tau} + \chi + \delta\chi^3 = \sin(\nu t)$$

(11)

Equation (11) is a harmonically forced Duffing equation [10]. A numerical solution to the problem is given in the following.

Taking, $\delta = 0.25$ (weak non-linearity) and $\beta = 0.10$ (light damping) the response can be obtained with the aid of MAPLE® (initial conditions are set to $\chi = 0$ and $d\chi/d\tau = 0$, see appendix for worksheet).

The goal is to illustrate the "jump phenomena", where the steady state response amplitude undergoes a jump when the frequency of the excitation approaches the linear resonant frequency ($\nu = 1$). The response is obtained for several values of the excitation frequency and the steady state values are plotted versus the frequency ratio ν . Figure 9 through Figure 12 illustrate the responses and the steady state values for excitation frequencies close to the linear resonance frequency. Note the increase of the steady state values and the sudden drop ("jump") at $\nu = 1.5$.

Figure 13 depicts the frequency response for the system obtained numerically. A "jump" is clearly observed in the vicinity of $\nu = 1.5$.

Also included in the figure is the frequency response for the system when $\delta = 0$ (linear system, response peaks at the resonant level $\nu = 1$):
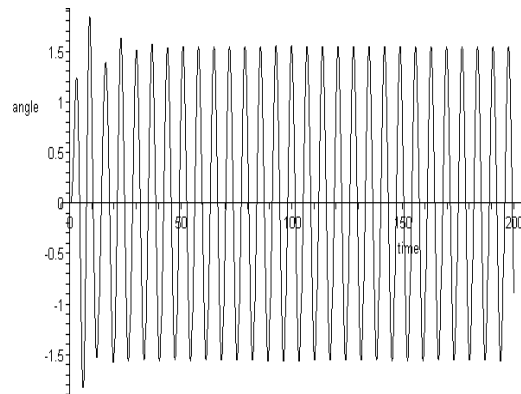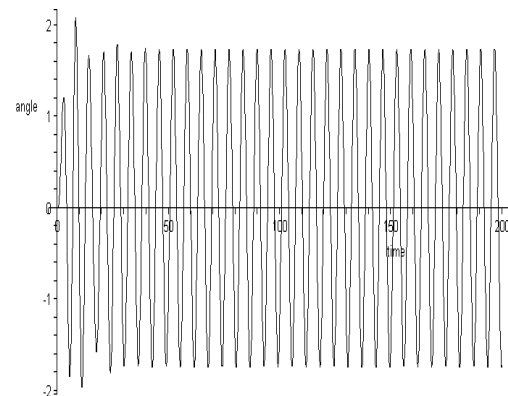

Figure 9. response for $\nu = 0.9$.
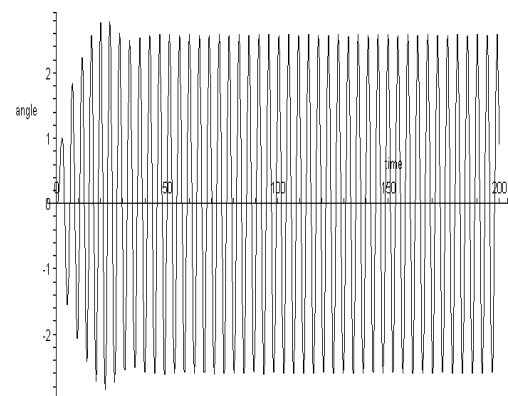

Figure 10. response for $\nu = 1$.

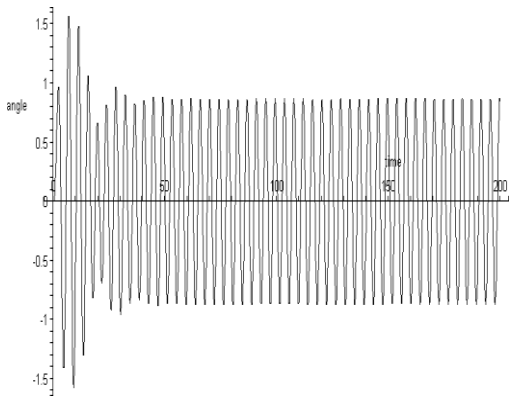Figure 11. response for $\nu = 1.4$.

Figure 12. response for $\nu = 1.5$.

Students should notice the "bending" of the resonant peak in the case of a non-linear system. Here the peak is bent to the right since the non-linearity is of a hardening type (in the case of a softening non-linearity the peak would bend to the left). No steady state values were obtained for the unstable solutions highlighted in the figure. These are unstable stationary solutions and will not be observed in practice.
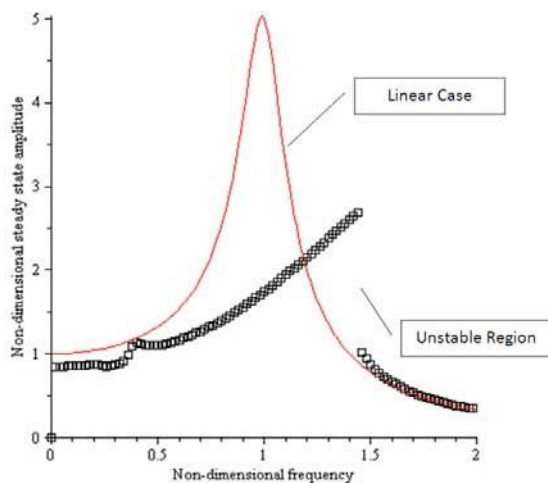

Figure 13. Numerically obtained frequency response.

Conclusions

In many institutions the required introductory courses now expose students to mathematics software such as MATLAB®, MAPLE®, etc. Advantage can be taken of this in beginning mechanics courses to explore some hitherto intractable problems which shed light on interesting and important dynamical phenomena. The examples presented were: (i) A comparison between some non-linear systems and their linearized counterparts; (ii) A simple pendulum with an oscillating support, illustrating parametric resonance. Depending on the parameter values of the system instabilities can occur (parametric resonance). This is shown numerically and confirmed with an available analytic expression. (iii) Forced harmonic motion of a non-linear hardening spring-mass system. The numerical simulation of the response illustrates a "jump phenomena" in which the steady state amplitude undergoes a jump in passing through frequencies close to the linear resonance frequency.

References

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2. www.ptc.com/products/mathcad/.

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4. A. Mazzei, Integrating Simulation Software into an Undergraduate Dynamics Course: A Web-Based Approach, *Proceedings of the 2003 American Society for Engineering Education Annual Conference & Exposition, Nashville - TN*, 2003.

5. www.mscsoftware.com/Contents/Products/ CAE-Tools/Adams.aspx.

6. P. Gharghouri, Integrating a Computer Algebra Software into Engineering Curriculum: Problem and Benefits, *Proceedings of the 1998 American Society for Engineering Education Annual Conference & Exposition, Seattle - WA*, 1998.

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8. J. Kiusalaas, *Numerical Methods in Engineering with Matlab®*, Cambridge University Press, 2005.

9. V. V. Bolotin, *The Dynamic Stability of Elastic Systems*, San Francisco, California, Holden-Day, Inc., 1964.

10. J. J. Thomsen, *Vibrations and Stability*, McGraw-Hill, 1997.

Appendix

MAPLE ® Worksheets

```
# Maple: Euler's method, runge-kutta
and linear approximation
restart:with(linalg):with(plots):with(D
Etools):
eq01:=diff(theta(t),t$2)+sin(theta(t));
eq02:=diff(theta(t),t$2)+(theta(t));
initial_angle:=evalf(convert(45*degrees
,radians));
 initial_velocity:=0;
ic1:=theta(0)=initial_angle,D(theta)(0)
=initial_velocity;
ic2:=theta(0)=initial_angle/10,D(theta)
(0)=initial_velocity;
soln02_1:=evalf(dsolve({eq02,ic1}));
soln02_2:=evalf(dsolve({eq02,ic2}));
soln01_1:=dsolve({eq01,ic1},numeric,out
put=procedurelist);
soln01_2:=dsolve({eq01,ic2},numeric,out
put=procedurelist);
fig01:=odeplot(soln01_1,t=0..50,numpoin
ts=500,labels=["time","angle"],labeldir
ections=[horizontal,vertical],color=bla
ck,style=line,symbol=circle):
fig02:=odeplot(soln01_2,t=0..50,numpoin
ts=500,labels=["time","angle"],labeldir
ections=[horizontal,vertical],color=blu
e,style=point,symbol=box):
 fig03:=DEplot(eq02,theta(t),t =
0..50,[{ic1}],numpoints=500,labels=["ti
me","angle"]):
 fig04:=DEplot(eq02,theta(t),t =
0..50,[{ic2}],numpoints=500,labels=["ti
me","angle"]):
 #
 display(fig01,fig03);
 display(fig02,fig04);
 display(fig01,fig02);
 # Simple Euler's Method - non-linear
 h:=.01;
y2(0):=initial_velocity;y1(0):=initial_
angle/10;
 number:=2500;
```

```
 for i from 0 to number do:
y1(i+1):=y1(i)+h*y2(i);y2(i+1):=y2(i)+h
*(-sin(y1(i)));
c1[i+1]:=[i*h,y1(i+1)];c2[i+1]:=[i*h,y2
(i+1)];
 end do:
 mat01 := array(1..number):mat02 :=
array(1..number):
for j from 1 to number do
mat01[j] := c1[j]: mat02[j]:= c2[j]:
end do:
#print(mat01);#print(mat02);
fig05:=pointplot(mat01,color=black,labe
ls=["Non-dimensional time","Amplitude -
rad"],labeldirections=[horizontal,verti
cal],symbol='box',symbolsize=5):display
(%);
#fig06:=pointplot(mat02,color=black,lab
els=["Non-dimensional time","Amplitude
rad"],labeldirections=[horizontal,verti
cal],symbol='box',symbolsize=5):display
(%);
 display(fig04,fig05);
 # Simple Euler's Method - linear
 h:=.01;
y2(0):=initial_velocity;y1(0):=initial_
angle/10;
 number:=2500;
 for i from 0 to number do:
y1(i+1):=y1(i)+h*y2(i);y2(i+1):=y2(i)+h
*(-(y1(i)));
c1[i+1]:=[i*h,y1(i+1)];c2[i+1]:=[i*h,y2
(i+1)];
 end do:
 mat03 := array(1..number):mat04 :=
array(1..number):
for j from 1 to number do
mat03[j] := c1[j]: mat04[j]:= c2[j]:
end do:
#print(mat03);#print(mat04);
fig07:=pointplot(mat03,color=black,labe
ls=["Non-dimensional time","Amplitude -
rad"],labeldirections=[horizontal,verti
cal],symbol='box',symbolsize=5):display
(%);
 display(fig04,fig07);


 # Pendulum on an oscillating support
 restart:with(linalg):
eq01:=diff(g(x),x$2)+c*diff(g(x),x)+(nu
0^2-nu^2*q*cos(nu*x))*g(x);
g(x):=a1*sin(nu*x/2)+b1*cos(nu*x/2);
 eq01;
 eq1:=combine(eq01,trig);
eq1:=sort(eq1,[sin(nu*x/2),cos(nu*x/2)]
);
eq1a:=collect(eq1,[sin(nu*x/2),cos(nu*x
/2)]);
 coef01:=coeff(eq1a,sin(nu*x/2),1);
```

```
coef02:=coeff(eq1a,cos(nu*x/2),1);
m1:=collect(coef01,[a1,b1]);
m2:=collect(coef02,[a1,b1]);
matriz:=matrix(2,2,[coeff(m1,a1,1),coef
f(m1,b1,1),coeff(m2,a1,1),coeff(m2,b1,1
)]);
equacao:=det(matriz);
raizes:=solve(equacao,q);
c:=.01;nu0:=1;
raizes;
raizes[1];
raizes[2];
#raizes[3];
#raizes[4];
plot([raizes[1],raizes[2]],nu=0..5,0..1
0);
plot([raizes[1],raizes[2]],nu=1.5..2.5,
0. 0.4,labels=["nu","q"],color=black);
restart:with(linalg):with(DEtools):with
(plots):
eq01:=diff(g(x),x$2)+c*diff(g(x),x)+(nu
0^2-nu^2*q*cos(nu*x))*g(x);
c:=0.01;q:=0.2;nu0:=1;nu:=2;
eq01;
sol001:=dsolve({eq01,g(0)=0.1,D(g)(0)=0
},{g(x)}, type=numeric,
method=gear,output=procedurelist);
odeplot(sol001,[x,g(x)],0..50,numpoints
=1000,color=black,labels=["time","ampli
tude"],labeldirections=[horizontal,vert
ical]);
c:=0.01;q:=0.2;nu0:=1;nu:=1.6;
eq01;
sol001:=dsolve({eq01,g(0)=0.1,D(g)(0)=0
},{g(x)}, type=numeric,
method=gear,output=procedurelist);
odeplot(sol001,[x,g(x)],0..200,numpoint
s=1000,color=black,labels=["time","ampl
itude"],labeldirections=[horizontal,ver
tical]);
```

Duffing equation

```
restart:
with(linalg):with(plots):with(DEtools):
with(plottools):
eq01:=(diff(x(t), `$`(t,
2)))+2*beta*(diff(x(t),
t))+x(t)+delta*x(t)^3-sin(nu*t);
delta:=0.25;beta:=0.1;
eq01;
nu:=0;n:=400;number:=100;
for i from 1 to number do
```

```
eq01:
sol001:=dsolve({eq01,x(0)=0,D(x)(0)=0},
{x(t)}, type=numeric,
method=gear,output=listprocedure):xt:=e
val(x(t),sol001):
xxt:=[seq(abs(xt(j)),j=300..n)]:f1:=max
((xxt)):p[i]:=[nu,f1]:
odeplot(sol001,[t,x(t)],0..n,numpoints=
2500,color=black,labels=["non-
dimensional
time","amplitude"],axes=boxed);
nu:=nu+0.02:
end do;
mat01 := array(1..number):
for j from 1 to number do mat01[j] :=
p[j] end do:
print(mat01);
fig01:=pointplot(mat01,color=black,labe
ls=["Non-dimensional
Frequency","Amplitude"],labeldirections
=[horizontal,vertical],symbol='box',sym
bolsize=15):display(fig01);
eq02:=(diff(x(tau), `$`(tau,
2)))+2*beta*(diff(x(tau),
tau))+x(tau)+delta*x(tau)^3-
sin(omega/omega0*tau);
amp:=1/(sqrt((1-
(omega/omega0)^2)^2+(2*beta*omega/omega
0)^2));
omega0:=1;delta:=0;beta:=0.10;
eq02;
amp;
fig02:=plot(amp,omega=0..2):
plots[display]([fig01,fig02],labels=["N
on-dimensional frequency","Non-
dimensional steady state
amplitude"],labeldirections=[horizontal
,vertical]);
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Biographical Information

Arnaldo Mazzei is an Associate Professor of Mechanical Engineering at Kettering University. He received his Ph.D. in Mechanical Engineering from the University of Michigan in 1998. He specializes in dynamics and vibrations of mechanical systems and stability of drivetrains with universal joints. His current work relates to stability of drivetrains, finite element analysis and CAE. He is a member of ASME, ASEE and SEM.