# MATHEMATICAL MODELING AND ANALYSIS: AN EXAMPLE USING A CATAPULT

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**Abstract**

In our mathematical modeling sequence we spend a lot of time covering techniques and methods to accomplish mathematical modeling. Rarely do students get an opportunity to perform all the different steps in the modeling process. We are able to have the students experience these steps with our "modeling the catapult" experiment. The students build a mathematical model based upon certain assumptions, collect their own data, and use their model to hit a target at some predetermined distance during class. In our courses at the Naval Postgraduate School (NPS), we have had over 90% of the student teams hit the target with the first shot and 95% hit the target within two shots.

Keywords: mathematical modeling, multivariable regression, catapult

## Introduction and Background to Mathematical Modeling

We teach a three course sequence of mathematical modeling to our students in our department at the Naval Postgraduate School (NPS). Although many had a calculus course prior to our modeling course the course requirement to begin the sequence is only college algebra. We must build upon those skills in covering modeling topics in discrete modeling, stochastic modeling, and decision theory & game theory modeling.

This article discusses an experiment that we do that enables the student to experience the entire modeling process (we discuss the process later). In our first modeling course, we discuss methods of linear regression and model adequacy through residual plot analysis. In our second modeling course, we discuss some more advanced model fitting methods and illustrate them in class.

Our sequence is based upon mathematical modeling. We use the modeling process explained in Chapter 2 of A First Course in Mathematical Modeling [5]. We want the students to know the process is not exact, as we move from reality into the mathematical world. We try to model the reality but we use mathematics to build simple models, mathematics to solve the model, and then we attempt to interpret the mathematical results back into reality. The mathematical model allows us to use mathematical operations or algorithms to reach mathematical conclusions about the model as illustrated in the Figure 1 [3-5] that we relate to the real world. We can observe real-world behavior through the data from real world systems as an iterative process. We simplify the real world system into objects that we can model mathematically. If the results do not match reality as closely as we would want then we enhance the model and try again.

## Models and real-world systems

Students need to understand what constitutes a system. We define a system as any group of objects joined by interaction or interdependence. Combat between opponents, the United States economy, a bass fish population growing in a small lake, a communications satellite orbiting the earth, delivering U.S. mail via mail routes, locations of service facilities, weapons systems, and catapults are a few examples of a system. The person modeling the system is interested perhaps in either understanding how it works to find flaws or predict the outcomes from its intended use.

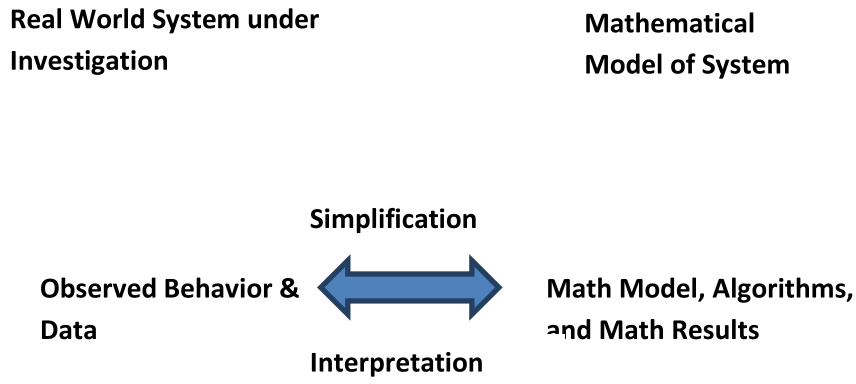


Figure 1. Modeling real world systems with mathematics[5].

A basic technique used in constructing a mathematical model of some system might be a combined mathematical-physical analysis. For example, we might start with distance, rate and time and use D=RT or force, mass, and acceleration and use F=MA if they apply. Then we reason logically to obtain conclusions.

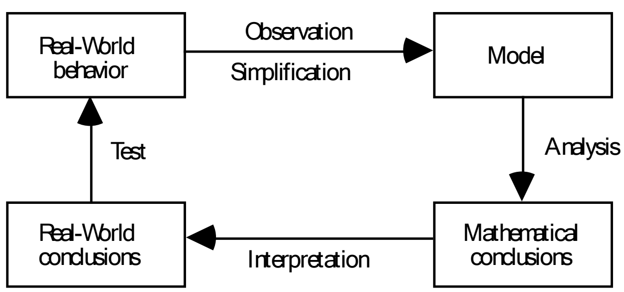


Figure 2. In reaching conclusions about a real-world behavior, the modeling process is a closed system [5].

According to our books [3,4] in mathematical modeling we use Figure 2 to suggest how we can obtain real-world conclusions from a mathematical model. First, we brain storm and through observations of the system we identify the factors that seem to be involved in the behavior that we want to model. Often we have collected data or we go out and collect the data that represents the system (or systems) for which we have interest. Next, we plot the data and look for recognizable patterns. If there is a pattern in the data we consider model fitting techniques using regression techniques in order to create a rough model of the system's behavior. We might use the model to explain or predict system behavior. Thus, we apply mathematical reasoning that leads to conclusions about the model. These conclusions apply to our mathematical model (and may or may not apply to the actual real-world system in question). We might compare the model's results with reality to determine how well our model works. If the model is reasonably valid, the results of the model can be used to draw interpretations about the real-world behavior from the model's conclusions. In summary, in our text we have the following five step procedure for investigating real-world behavior [5]:

1. Through observation, identify the primary factors involved in the real-world behavior, possibly making simplifications.
2. Conjecture tentative relationships among the factors.
3. Apply mathematical reasoning to the resultant "model."
4. Interpret the mathematical conclusions in terms of the real-world system.
5. Test the model conclusions against real-world observations.

Modeling is creative and these activities give us insights into the mathematical aspects of the problem and reality.

Often our own time table to obtain adequate results limits the continuation of model improvement and model refinement. Thus, the better the initial model the better off the students might be in their final model analysis. Time is certainly a factor as we describe in our catapult example.

In previous published uses of the catapult, the approach has been more of an experimental design using either fractional factorial design or Taguchi Design methods [1, 6, 7, 8 ,9]. Our current students have not been exposed to advanced statistical design but only college algebra so we only use a regression modeling approach.

## MODEL CONSTRUCTION

Model construction outlines a process to help construct mathematical models. We present an eight step approach modified from several modeling sources, such as illustrated in [5] and COMAP's Mathematical Contest in Modeling's format requirement (www.COMAP.com). These eight steps are summarized in Figure 3. These steps act as a guide for thinking about the problem and getting started in the modeling process.

It is these eight steps that our students experience with the catapult experiment. It is the only time in our three course sequence that the student gets an opportunity to experience the modeling process from start to finish, as we will describe.

### The Catapult Experiment

We show a typical catapult in Figure 4. The catapult, known as the STATPULT (www.ncmrcompany.com), can be obtained from the NCMR Company. Readers may visit their web site to check on items that are available (specifically, visit their website at www.ncmrcompany.com). Although the company's STAPULT was created for statistical design studies, we have been using it for mathematical modeling since about 1990.

| |
|---|
| Step 1. Understand the problem or the question asked. |
| Step 2. Make simplifying assumptions. |
| Step 3. Define all variables. |
| Step 4. Construct a model. |
| Step 5. Solve and interpret the model. |
| Step 6. Verify the model. |
| Step 7. Identify the strengths and weaknesses of your model. |
| Step 8. Implement and maintain the model for future use. |

Figure 3. Mathematical Modeling Process [5] ( www.comap.com).


Figure 4. The Catapults used in class.

There are many settings (that can be used with the catapult) that we call variables. These include:

- **Elevation:** is the ability to tilt the entire unit.
- **Stop Position:** there are six positions to stop the arms going forward.
- **Stationary Arm Position**: there are three different places that you can attach the elastic rubber band to the stationary up right.
- **Pivot Arm Position:** there are three locations to attach the elastic rubber band to the pivot arm.
- **Ball Seat Position:** there are three different locations that you can place the ball (rubber, plastic, or whiffle) for throwing from the pivot arm.
- **Pull Angle Position**: these are the angles for pulling the pivot arm back to before release from 90-190 degrees.
- **Ball Type:** different balls: rubber, plastic, and whiffle.

We typically do not initially model with all these factors. We start with two balls that the students choose from among the three available balls (rubber, plastic, or whiffle), the pull angle position 90-190 degrees, and the stop position. The ball is actually a non-factor that we refer to as noise. Our students do keep track of the different balls as they feel certain balls travel further than other balls.

## The Catapult Modeling Process

We want the students to experience the entire eight step modeling process in our course. First, we pass around the catapult and ask students questions about it. We even have a student come forward and, after placing a coffee cup down range, ask the student to pick a ball and hit the target. On their initial shot they do not usually come close to hitting the target. This enables us to obtain our problem identification statement (PID).

PID: *Predict the distance the ball travels being shot from the catapult*.

We begin making simplifying assumptions about the catapult in order to help us with the modeling. Our students have finished a block on modeling. Our students have finished a block on model fitting and regression, having built simple polynomial models and at least one multivariable regression model usually on the cost of a home as a function of square feet, the number of bedrooms, and the number of bathrooms.

We list all the factors that affect the distance the ball travels on the board through an interactive session with the students. Then, we begin simplifying these factors leaving distance, stop arm position, and angle as our remaining key variables. All other variables are assumed constant and allowed to be consumed as part of the constant in the model. In this way a constant makes sense.

Students are broken into teams. We usually have team sizes of at least four students. Additional equipment items include a measuring tape and duct tape (to secure the catapult into position). Teams are encouraged to either place the catapult on the ground or on a table edge of a long classroom flat table. Each team chooses two stop settings from the six possible positions (see Figure 5) and two angles (see Figure 6) that will work for their stop positions. Using each of the balls, the students are required to fire two shots at each setting. This will give them sixteen sets of data points.


Figure 5. Stop Positions for the catapult arm.


Figure 6. Possible Pull Angles.

## Model Building

The mathematical model that we want to build is given in equation (1):

$$Distance = a_o + a_1\ Stop\ Position + a_2\ Angle + a_3\ Angle*Stop\ Position. \quad (1)$$

The *Angle *Stop Position* allows us to consider an interaction effect within the model. We take some time to explain and discuss this concept with the students. In our course, it is not the first time that we have introduced an interaction effect. We introduced these interaction effects in their first modeling course when we covered Lanchester equations in discrete dynamical systems [2].

## Data Collection

The actual data collection has always been an experience in itself. Even after discussing the phenomena of the data collection process, the students still fall into the traps. Students cannot agree on exactly (precisely) where the ball hits in order to measure the distance. Two students see the event differently. Students are told to reach an agreement where the ball hit and measure the distance. They want to discuss ways to fix this such as using carbon paper or a sandbox where the ball will make marks.

Students think the distance that the ball travels each time (with the same setting and ball) must be the same—they want to have “do overs” if it is not the same. This encourages good discussion of data collection and the possible costs involved in such collection of data and “do-overs”. Both time and money might prevent the possibility of “do-overs” as the cost may be too high to have any “do-overs”. An example of a possible collection of data is shown in Figure 7.

We use Excel to obtain the multiple regression model. Excel is our software of choice since all our graduates will have Excel on their computers after graduation. Figure 8 exhibits our output for our data and our model shown in equation (1). From our Excel output, we write the model we obtained. If you have the student teams place their models on a board, they can see that each team has a different model. With our data the model is given in equation (2).

| Distance | Stop | Angle | Stop*Angle |
|---|---|---|---|
| 56 | 1 | 150 | 150 |
| 38 | 1 | 150 | 150 |
| 48 | -1 | 150 | -150 |
| 38 | -1 | 150 | -150 |
| 55 | 1 | 150 | 150 |
| 39 | 1 | 150 | 150 |
| 58 | -1 | 150 | -150 |
| 39 | -1 | 150 | -150 |
| 111 | 1 | 170 | 170 |
| 88 | 1 | 170 | 170 |
| 98 | -1 | 170 | -170 |
| 88 | -1 | 170 | -170 |
| 112 | 1 | 170 | 170 |
| 90 | 1 | 170 | 170 |
| 108 | -1 | 170 | -170 |
| 85 | -1 | 170 | -170 |

Figure 7**.** Sample collected data.

$$Distance = -484.917+21.333*Stop+3.40833*Angle-0.14167*Stop*Angle. \quad (2)$$

We have now completed the first five steps of the modeling process. The next steps have the students complete the verification and validation process of modeling.

## Verification and Validation

Students bring their models to class the next meeting after having emailed the raw data to the instructor. We do not want to give a distance for a target where a team has as an actual data point. We select a distance, for example, 100 inches, and place a coffee mug centered at that distance. Now each team calculates the stop position and the angle from their model to use to hit the target with their catapult. Many teams also desire to select the ball they want to shoot to hit the target

SUMMARY OUTPUT

| Regression Statistics | |
|---|---|
| Multiple R | 0.939977 |
| R Square | 0.883557 |
| Adjusted R Sc | 0.854446 |
| Standard Erro | 10.74806 |
| Observations | 16 |

ANOVA

| | df | SS | MS | F | Significance F |
|---|---|---|---|---|---|
| Regression | 3 | 10518.69 | 3506.229 | 30.35149 | 6.93579E-06 |
| Residual | 12 | 1386.25 | 115.5208 | | |
| Total | 15 | 11904.94 | | | |

| | Coefficients | tandard Erro | t Stat | P-value | Lower 95% | Upper 95% | ower 95.0% | Jpper 95.0% |
|---|---|---|---|---|---|---|---|---|
| Intercept | -337.063 | 43.07613 | -7.82481 | 4.71E-06 | -430.9173351 | -243.208 | -430.917 | -243.208 |
| Stop | -15.3125 | 43.07613 | -0.35548 | 0.728404 | -109.1673351 | 78.54234 | -109.167 | 78.54234 |
| Angle | 2.55625 | 0.268702 | 9.513343 | 6.12E-07 | 1.970799624 | 3.1417 | 1.9708 | 3.1417 |
| Stop*Angle | 0.10625 | 0.268702 | 0.39542 | 0.699472 | -0.479200376 | 0.6917 | -0.4792 | 0.6917 |

RESIDUAL OUTPUT

| Observation | dicted Dista | Residuals |
|---|---|---|
| 1 | 47 | 9 |
| 2 | 47 | -9 |
| 3 | 45.75 | 2.25 |
| 4 | 45.75 | -7.75 |
| 5 | 47 | 8 |
| 6 | 47 | -8 |
| 7 | 45.75 | 12.25 |
| 8 | 45.75 | -6.75 |
| 9 | 100.25 | 10.75 |
| 10 | 100.25 | -12.25 |
| 11 | 94.75 | 3.25 |
| 12 | 94.75 | -6.75 |
| 13 | 100.25 | 11.75 |
| 14 | 100.25 | -10.25 |
| 15 | 94.75 | 13.25 |
| 16 | 94.75 | -9.75 |

Figure 8. Excel output.

The teams know the target distance is 100 inches as given and might assume the stop setting is in position # 1. Thus, they might calculate the angle as follows:

100 = -484.917 + 21.333*(1) + 3.40833*Angle - 0.14167*(1)*Angle 3.2666 Angle=563.584. They solve for the Angle, Angle = 172.5292353 degrees

Refer to Figure 6 for the angle and note that it is in degree increments. Students need to decide how they are to handle the 0.5292 of a degree.

The students then set up their catapult, center it on the target, use their settings, and prepare to "fire for effect" at the coffee mug.

At NPS. we have had over a 90% success rate of student teams hitting the target on the first shot. We note this because at two other universities where we have done this modeling, no team hit the target (they did came close) on the first shot. Since we are using a coffee mug, teams have tried to actually have the ball land inside the mug. The teams that did not hit the target come very close and usually the miss was attributed to numerical rounding that their team has done.

Only once did we have a team totally unable to hit or even come close to the target. We mention this because of why it happened. In the course of two separate classes, the second class felt the rubber band was too frayed and decided to change it. When the first class took their catapults, the team that used that catapult was unaware of the change made by the other team. The first team's model was no longer valid for the catapult with a new rubber band. The system's objects had been modified changing the characteristics of the system. After we discovered this and put the original rubber band back in place the team did hit the target. We use this as a teaching example during the exercise.

## Strengths and Weaknesses of the Model

Self-reflection of the modeling is always a worthwhile endeavor. Teams are asked to discuss what about their model is good and what could be improved if they could do the process over. The ideas presented by the students also generate some interesting discussions.

## Conclusion

Students take away a real appreciation for the power and limitation of the modeling process. For example, we discuss if the model can be used for moving targets or how the model might be modified for moving targets. We notice that students feel like they better understand the modeling process and after hitting the target they appreciate the results of the modeling process. Often many student teams stay after class to try to get the ball to land directly into the coffee cup that we use as a target.

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## Biographical Information

Dr. William P. Fox is a professor in the Department of Defense Analysis at the Naval Postgraduate School. He received his BS degree from the United States Military Academy at West Point, New York, his MS from the Naval Postgraduate School, and his Ph.D. from Clemson University. Previously he has taught at the United States Military Academy and Francis Marion University, where he was the chair of the mathematics department for eight years. He has many publications including books, chapters, journal articles, conference presentations, and workshops. He directs several math modeling contests through COMAP. His interests include applied mathematics, optimization (linear and nonlinear), mathematical modeling, statistical models for medical research, and computer simulations. He is currently the President of the Military Application Society in INFORMS.