MATHEMATICAL MODELING OF THE ANALYTICAL HIERARCHY PROCESS USING DISCRETE DYNAMICAL SYSTEMS IN DECISION ANALYSIS

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Abstract

In our redevelopment of our three course sequence in mathematical modeling for decision making, we decided to add some additional topics for decision making. One such topic was the Analytical Hierarchy Process developed by Saaty in the 1970's. In looking at the process required to solve such problems there were many concepts well beyond the capabilities of our students, whose mathematical background was college algebra. The ideas of matrices, applications of matrix algebra, computational matrix algebra, and eigenvectors were a bridge too far. We teach discrete dynamical systems in our first course as a method to obtain a discrete Lanchester's equation for combat analysis that we develop into models for Insurgency operations. We found that the ideas from AHP fit into DDS very nicely. We are able to reuse our DDS and Lanchester's concepts of stability and parity to solve the complex decisions using the principals of the pairwise comparison and normalizations to build sets of discrete dynamical systems of equations. This paper illustrates this concept that our students, future decision makers in high positions, can understand.

Keywords: AHP, DDS, stable equilibrium values

Introduction

We teach a three course sequence in mathematical modeling for decision making for our students in our interdisciplinary department of defense analysis. Our three course sequence goal is to prepare these students to become *competent, confident problem solvers* for the 21st century. It would be helpful to understand the context by describing these three courses. The first course covers deterministic models for decision making. Topics include discrete dynamical systems, model fitting with least squares, and linear optimization. Another course covers decision theory and game theory techniques. The last course covers probabilistic modeling with data analysis, introductory probability concepts, a few discrete and continuous distributions, hypothesis testing, advanced regression techniques, and an introduction to networks. It is in this last course that we desire to add an introduction to decision making with AHP.

The **Analytic Hierarchy Process (AHP)** is a structured technique for organizing and analyzing complex decisions. It was developed by Thomas Saaty (1977) and is based on two main sound principles: mathematics and psychology. It is used and taught worldwide; we know it is taught extensively in China whose thousands of modeling teams use it quite often in the Mathematical Contest in Modeling.

In many articles and publication, Saaty (2008)[4] and others (Hass et al.)[3] show how AHP has been used as an application in group decision making around the world in a wide variety of decision situations, in fields such as government, business, industry, healthcare, and education. Since our students are primarily government and military personnel AHP fits nicely into the decision knowledge base toolkit.

AHP Background

We only desire to briefly discuss the elements in the framework of AHP. AHP can be

described as a method to decompose a problem into sub-problems. In most decisions, the decision maker has a choice among several to many alternatives. Each alternative has a set of attributes or characteristics that can be measured, either subjectively or objectively. The attribute elements of the hierarchal process can relate to any aspect of the decision problem—tangible or intangible, carefully measured or roughly estimated, well- or poorly-understood—anything at all that applies to the decision at hand.

In its simplest sense we can state that in order to perform AHP we need an objective, a set of alternatives, each with attributes to compare.

Objective Statement ← This is the decision desired

Alternatives: *1, 2, 3, …, n*

For each of the alternatives there are attributes to compare.

Attributes: $a_1, a_2, …, a_m$

Once the hierarchy is built, the decision maker(s) systematically evaluate its various elements pairwise (by comparing them to one another two at a time), with respect to their impact on an element above them in the hierarchy. In making the comparisons, the decision makers can use concrete data about the elements, but they typically use their judgments about the elements' relative meaning and importance. It is the essence of the AHP that human judgments, and not just the underlying information, both can be used in performing the evaluations.

The AHP converts these evaluations to numerical values that can be processed and compared over the entire range of the problem. A numerical weight or priority is derived for each element of the hierarchy, allowing diverse and often incommensurable elements to be compared to one another in a rational and consistent way. This capability distinguishes the AHP from other decision making techniques.

In the final step of the process, numerical priorities are calculated for each of the decision alternatives. These numbers represent the alternatives' relative ability to achieve the decision goal, so they allow a straightforward consideration of the various courses of action.

Uses and Applications

While it can be used by individuals working on straightforward decisions, the Analytic Hierarchy Process (AHP) is most useful where teams of people are working on complex problems, especially those with high stakes, involving human perceptions and judgments, whose resolutions have long-term repercussions. It has unique advantages when important elements of the decision are difficult to quantify or compare, or where communication among team members is impeded by their different specializations, terminologies, or perspectives.

Decision situations to which the AHP can be applied include:

- Choice - The selection of one alternative from a given set of alternatives, usually where there are multiple decision criteria involved.
- Ranking - Putting a set of alternatives in order from most to least desirable
- Prioritization - Determining the relative merit of members of a set of alternatives, as opposed to selecting a single one or merely ranking them
- Resource allocation - Apportioning resources among a set of alternatives
- Benchmarking - Comparing the processes in one's own organization with those of other best-of-breed organizations
- Quality management - Dealing with the multidimensional aspects of quality and quality improvement

- Conflict resolution - Settling disputes between parties with apparently incompatible goals or positions

Using the Analytic Hierarchy Process

The procedure for using the AHP can be summarized as:

1. Model the problem as a hierarchy containing the decision goal, the alternatives for reaching it, and the criteria for evaluating the alternatives.
2. Establish priorities among the elements of the hierarchy by making a series of judgments based on pairwise comparisons of the elements. For example, when comparing potential real-estate purchases, the investors might say they prefer location over price and price over timing.
3. Synthesize these judgments to yield a set of overall priorities for the hierarchy. This would combine the investors' judgments about location, price and timing for properties A, B, C, and D into overall priorities for each property.
4. Check the consistency of the judgments.
5. Come to a final decision based on the results of this process.

These steps are more fully described below.

Step 1. Build the hierarchy for the decision.

Goal   Select the best product
Criteria   Price, Support, and Ease of Use
Alternatives: Product X, Product Y, and
                Product Z

Step 2. Judgments and Comparison.

Build a numerical representation using a 1-9 point scale in a pairwise comparison for the attributes criterion and the alternatives. The goal, in AHP, is to obtain a set of eigenvectors of the system that measures the importance with respect to the criterion. We can put these values into a table. See Table 1.

Step 3. Using all the eigenvectors combined in order to obtain a comparative ranking.

Since our students have had only college algebra, covering matrices, matrix operations, eigenvalues and eigenvectors is quite a stretch. However, we have previously covered discrete dynamical systems in our first course. Students build DDS models and found solutions numerically through iterations and graphically. They have a good grasp of the concept of stability. What they were not taught was the relationship between stable solutions and the use of eigenvalues and eigenvectors in the closed form solution. We take advantage of the concept of stable solutions to obtain our stable set of values.

Table 1.

Intensity of Importance	Definition
1	**Equal Importance**
3	**Moderate Importance**
5	**Strong Importance**
7	**Very Strong Importance**
9	**Extreme Importance**
2, 4, 6, 8	**For compromises between the above**
Reciprocals of above	**In comparing elements i and j** **- if i is 3 compared to j** **- then j is 1/3 compared to i**
Rationals	**Force consistency** **Measured values available**

We must first describe a little background on discrete dynamical systems. For more information on DDS (see Fox (2012)[1], Giordano et al (2008)[2]).

Discrete Dynamical Systems

In modeling the world around us, we are often interested in predicting the value of a variable at some time in the future. Perhaps it is a population, a real estate value, or the number of people with a communicative disease. For these situations, we learn how to model change using the paradigm shown in equation (1) below,

future value = present value + change (1)

Often, we wish to estimate a future value based on what we know now, the present value, and adding the change that has been carefully observed and modeled. We construct numerical and graphical solutions to help predict the long-term outcomes of the system under study. We use the following function notation:

$a(n+1) = $ *Future value*
$a(n) = $ *Present value*
$F(a(n)) = $ *change*

We provide one example to illustrate the use of DDS and stable equilibriums.

Example for Prescribing Digoxin

The drug, digoxin, is used in the treatment of heart disease. Doctors must prescribe an amount of medicine that keeps the concentration of digoxin in the blood stream above an effective level without exceeding a safe level. Given an initial dosage of 1.5 mg in the bloodstream, and the drug absorption in the body is assumed to be 0.5 mg/day.

We formulated the following model, $a(n+1)=0.5\,a(n),\ a(0)=1.5\ mg$

The doctor prescribing the medicine now needs to determine a "maintenance dosage", an amount to take each day to maintain an

Cases		
I	II	III
6	3	1.5
4.5	3	2.25
3.75	3	2.625
3.375	3	2.8125
3.1875	3	2.9063
3.0938	3	2.9531
3.0469	3	2.9766
3.0234	3	2.9883
3.0117	3	2.9941
3.0059	3	2.9971
3.0029	3	2.9985
3.0015	3	2.9993
3.0007	3	2.9996
3.0004	3	2.9998
3.0002	3	2.9999
3.0001	3	3
3	3	3
3	3	3
3	3	3
3	3	3
3	3	3
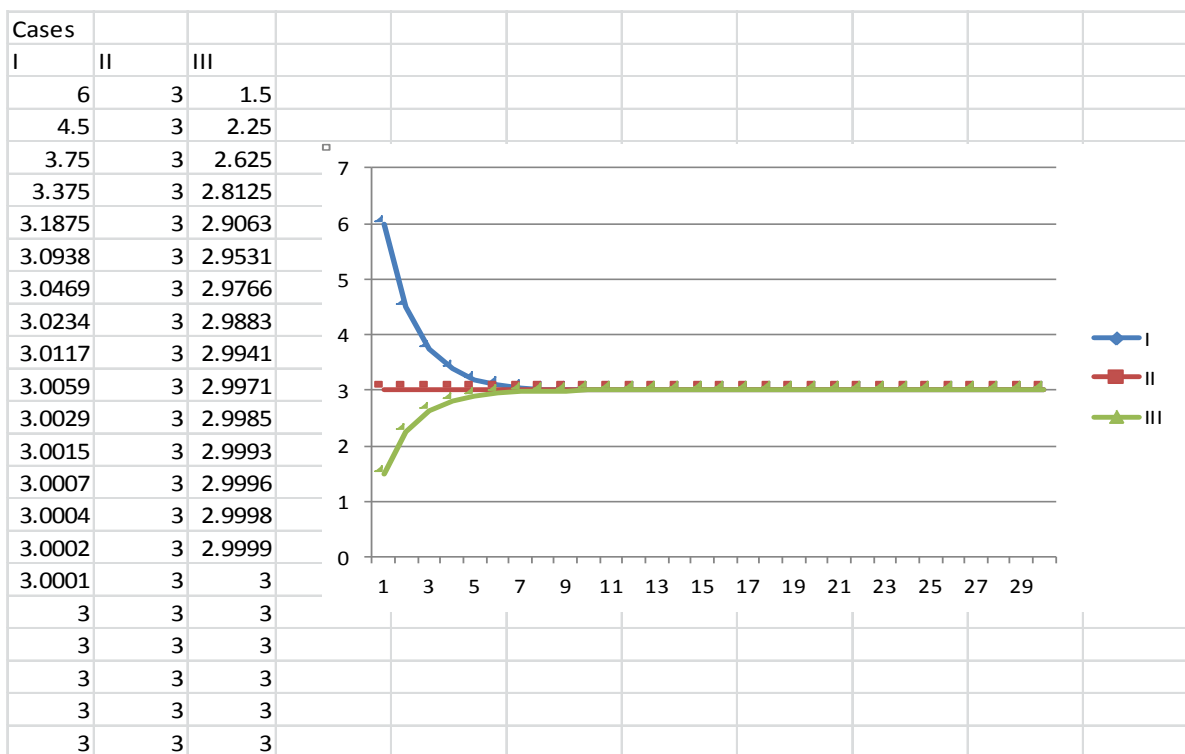
We note that all starting values converge to 3, so 3 is our stable equilibrium value.

appropriate level of digoxin in the bloodstream. The doctor assumes that keeping the dosage at 1.5 mg a day will be enough to keep the drug at a safe and effective level. Furthermore, let's see if an initial dosage that is greater or lesser than 1.5 makes a difference.

Our model now becomes $a(n+1)=0.5\ a(n) + 1.5, a(0)=1.5$.

We now iterate the difference equation $a(n+1)=0.5\ a(n) + 1.5$ from each of our starting values , 1.5, 3, & 6 and both iterate and graph the results.

Illustrative Example for AHP using DDS

Choosing a new product:

*state the objective---select a new product
*define criteria—price, support, ease of use
*pick alternatives—X,Y,Z

We use the 1-9 scale to establish relationships among the three alternatives with respect to the criterion. We will put the information into a table. How we get the values is either based upon subjective input, survey results, or actual data. The entries for XX, YY, ZZ are always 1. We measure XY, XZ, and YZ to completely fill out the top of the table using our 1-9 scale as defined earlier.

Criterion 1. Price

	X	Y	Z
X	1	5	7
Y		1	3
Z			1

Entries YX, ZX, and ZY are set to be the reciprocal of XY, XZ, and YZ.

	X	Y	Z
X	1	5	7
Y	1/5	1	3
Z	1/7	1/3	1

To better use DDS and the properties of stability, we normalize each column. This is done by dividing each coefficient by the sum of its column total. Column X total is 47/35, Column Y total is 19/3 and Column Z total is 11.

	X	Y	Z
X	35/47	15/19	7/11
Y	7/47	3/19	3/11
Z	5/47	1/19	1/11

This is where our methodology departs. We build a system of DDS equations from the columns.

$X(n+1)= 35/47\ X(n)+ 7/47\ Y(n) + 5/47\ Z(n)$

$Y(n+1)= 15/19\ X(n) + 3/19\ Y(n) + 1/19\ Z(n)$

$Z(n+1) = 7/11\ X(n) + 3/11\ Y(n) + 1/11\ Z(n)$

We initially set the initial conditions at $X(0)=1$, $Y(0)=0$, $Z(0)=0$ and iterate. By changing the initial condition to $X(0)=0$, $Y(0)=1$, $Z(0)=0$ and then $X(0)=0$, $Y(0)=0$, and $Z(0)=1$ we find the three stable equilibrium values for price.

X	Y	Z
1	0	0
0.744681	0.789474	0.636364
0.739829	0.746052	0.74705
0.741524	0.741192	0.742183
0.741545	0.741506	0.741493
0.741534	0.741536	0.74153
0.741533	0.741534	0.741534
0.741534	0.741533	0.741534
0.741534	0.741534	0.741534
0.741534	0.741534	0.741534
0.741534	0.741534	0.741534
0.741534	0.741534	0.741534
0.741534	0.741534	0.741534
0.741534	0.741534	0.741534
0.741534	0.741534	0.741534
0.741534	0.741534	0.741534

```
X          Y          Z
      0          1          0
0.148936   0.157895   0.272727
 0.16344   0.156866   0.162633
0.162375   0.162359   0.161574
0.162287    0.16233   0.162298
0.162295   0.162295     0.1623
0.162295   0.162295   0.162295
0.162295   0.162295   0.162295
0.162295   0.162295   0.162295
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0.162295   0.162295   0.162295
0.162295   0.162295   0.162295
0.162295   0.162295   0.162295
0.162295   0.162295   0.162295

X          Y          Z
      0          0          1
0.106383   0.052632   0.090909
0.096731   0.097082   0.090317
0.096101   0.096449   0.096244
0.096168   0.096163   0.096209
0.096172   0.096169   0.096171
0.096171   0.096171   0.096171
0.096171   0.096171   0.096171
```

Thus our three stable values are 0.741534, 0.162295, and 0.096171.

Criterion 2. Support

We begin by using our 1-9 scale to find pairwise comparisons between entries for our criterion support. We find YX=2, XZ=3, YZ=5 and

	X	Y	Z
X	1		2
Y	3	1	5
Z			1

Entries XY, ZX, and ZY are set to be the reciprocal of XY, XZ, and YZ.

	X	Y	Z
X	1	1/3	2
Y	3	1	5
Z	1/2	1/5	1

We sum the columns of X, Y, and Z respectively obtaining 9/2, 23/15, 8 so that we can normalize the columns.

	X	Y	Z
X	2/9	5/23	2/8
Y	6/9	15/23	5/8
Z	1/9	3/23	1/8

We can build our dynamical systems

$$X(n+1) = 2/9\ X(n) + 6/9\ Y(n) + 1/9\ Z(n)$$

$$Y(n+1) = 5/23\ X(n) + 15/23\ Y(n) + 3/23\ Z(n)$$

$$Z(n+1) = 2/8\ X(n) + 5/8\ Y(n) + 1/8\ Z(n)$$

We initially set the initial conditions at X(0)=1, Y(0)=0, Z(0)=0 and iterate. By changing the initial condition to X(0)=0, Y(0)=1, Z(0)=0 and then X(0)=0, Y(0)=0, and Z(0)=1 we find the three stable equilibrium values for support. Our three stable values are 0.222557, 0.65199, and 0.125452.

Criterion 3. Ease of Use

We begin, as before, using the 1-9 scale to relate X, Y, and Z. We find YX=2, ZX=3, and ZY=2.

	X	Y	Z
X	1		
Y	2	1	
Z	3	2	1

Now, we fill in as the reciprocal for XY, XZ, and YZ to obtain,

	X	Y	Z
X	1	0.5	1/3
Y	2	1	0.5
Z	3	2	1

We sum the columns so that we might normalize the columns. The sums are 6, 7/2, 11/6.

	X	Y	Z
X	1/6	1/7	2/11
Y	2/6	2/7	3/11
Z	3/6	4/7	6/11

We can build our dynamical systems

$X(n+1)= 1/6\ X(n)+ 2/6\ Y(n) + 3/6\ Z(n)$

$Y(n+1)= 1/7\ X(n) + 2/7\ Y(n) + 4/7\ Z(n)$

$Z(n+1) = 2/11\ X(n) + 3/11\ Y(n) + 6/11\ Z(n)$

We initially set the initial conditions at $X(0)=1$, $Y(0)=0$, $Z(0)=0$ and iterate. By changing the initial condition to $X(0)=0$, $Y(0)=1$, $Z(0)=0$ and then $X(0)=0$, $Y(0)=0$, and $Z(0)=1$ we find the three stable equilibrium values for the criterion ease of use. Iterating these we obtain our three stable values are 0.168102, 0.286635, and 0.545253.

Criterion Weights

Next we weight our criterion, price, support, and ease using the same process as before with our 1-9 scale.

	Price, P	Support, S	Ease, E
Price	1		
Support	3	1	
Ease	4	2	1

	Price, P	Support, S	Ease, E
Price	1	1/3	1/4
Support	3	1	1/2
Ease	4	2	1

The sums are 8, 10/3 and 7/4 so our normalized table is

	Price, P	Support, S	Ease, E
Price	1/8	1/10	1/7
Support	3/8	3/10	2/7
Ease	4/8	6/10	4/7

$P(n+1)= 1/8\ P(n)+ 3/8\ S(n) + 4/8\ E(n)$

$S(n+1)= 1/10\ P(n) + 3/10\ S(n) + 6/10\ E(n)$

$E(n+1) = 1/7\ P(n) + 2/7\ S(n) + 4/7\ E(n)$

We initially set the initial conditions at $P(0)=1$, $S(0)=0$, $E(0)=0$ and iterate. By changing the initial condition to $P(0)=0$, $S(0)=1$, $E(0)=0$ and then $P(0)=0$, $S(0)=0$, and $E(0)=1$ we find the three stable equilibrium values for the criterion ease of use. Iterating these we obtain our three stable values for criterion weights as 0.12766, 0.301418, 0.570922.

AHP Results

We still need to rank order our alternatives.

These values will be the ranks of our alternatives consistent with our criterion weights.

Criterion Weights	0.12766	0.301418	0.570922	
	X	Y	Z	
X	0.741535	0.222557	0.168102	
Y	0.162295	0.65199	0.286635	
Z	0.096171	0.125452	0.545253	

We multiply the criterion weights as a linear sum of the stable weights of the alternatives as follows:

X=0.12766*.741535+0.301418*0.162295+ 0.570922*0.096171=0.19594

Y=0.12766*0.222557+0.301418*0.65199+ 0.570922* 0.125452=0.296556

$$Z=0.12766*0.168102+0.301418*0.286635+0.5\ 70922*0.545253=0.419154$$

These are measured as ranking for our alternatives based on the stated criterion and the criterion weights. Z is ranked highest followed by Y and then X. Our AHP decision framework has ranked Z first.

Conclusion

We have shown that DDS can be used to find the eigenvectors of the attributes and alternatives elements in AHP. This methodology does not require knowledge of matrix algebra or eigenvectors. We have found the DDS is easily taught. Solutions found through iterations and graphs are more intuitive to students. They tend to understand the concepts of equilibrium and stable equilibrium as well.

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