FINITE ELEMENT ANALYSIS IN HEAT TRANSFER

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Introduction

The basic concept of finite element analysis is to take an object and divide it into a number of smaller objects. Then the relevant physical laws are applied to the small objects individually to produce a solution for the larger object. Carrying this process to smaller and smaller objects converges to the differential equation governing the process. However, solving the differential equation can be difficult. Analyzing a finite number of finite size objects can be simpler.

In this paper, the finite element method is applied to a pin fin that is transferring heat from a solid wall into a fluid that passes over the fin. In the first example below, the fin is taken to be a cylinder with an insulated tip. In the second example, the fin is again a cylinder, but there is convection from the tip. In the third example, the fin is conical in shape. In this case the mathematics of solving the differential equation is quite complicated, but the finite element approach is only slightly more complicated than the cylindrical fin case.

These examples are covered in a junior level course at Webb Institute. This is the students' first exposure to finite element analysis. They are quite familiar with Mathcad software, so this tool is used to implement the solution.

Fin Geometry

Figure 1 below shows a cylindrical fin attached to a flat wall. The fin is 5 centimeters long with a diameter of 1 centimeter. Made of aluminum, the fin has a thermal conductivity of 237 W/(mºC). It is attached to a wall that is maintained at 300ºC, and it protrudes into a fluid maintained at 15ºC, where the convection coefficient is 150 W/(m^{2}ºK). It is assumed that the tip of the fin is insulated. Answers include the temperature distribution along the length of the fin and the rate of heat transfer into the fluid.

In order to use the finite element method, the fin is subdivided axially into six elements. The element adjacent to the wall is 0.5 centimeters in length, and the element at the right end is also 0.5 centimeters in length. Each of the four interior elements is 1.0 centimeters in length.

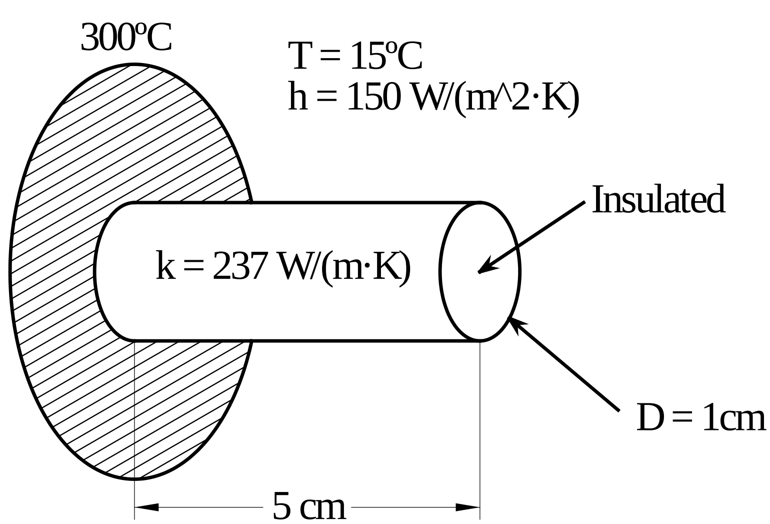

Figure 1: Cylindrical Fin.

Figure 2 below shows the cross-section view of the situation.

Fin Heat Transfer

Each element above is assumed to be at uniform temperature, and the temperature is assumed to change abruptly at the dashed lines that indicate the boundaries between elements.

An interior element (elements 1 through 4) has three heat flows: Heat flows in at the left face of the element from the previous element by conduction. Heat flows out at the right face of the element into the next element by conduction, and heat flows from the exposed surface of the element into the surrounding fluid by convection.

Element 0 is assumed to be at the known temperature of the wall, 300ºC. Element 5 has heat flowing in at the left face by conduction and heat flowing into the fluid from its exposed surface by convection. There is no heat transfer from the insulated end. Once the finite element model has been created, this end condition is easily modified to provide for convection at the end.

The well-known equations governing conduction and convection are

$$q_{conduction} = -k \cdot A_{cs} \frac{dT}{dx}$$

and

$$q_{convection} = h \cdot A_{surface} \left(T - T_{fluid}\right)$$

In finite element analysis the derivative is approximated by ΔT/Δx, so

$$q_{conduction} = -k \cdot A_{cs} \frac{\Delta T}{\Delta x}$$

These equations were used to construct an energy balance for each element with the exception of element 0. For example the energy balance for element 1, an interior element, is

$$-k \cdot A_{cs} \frac{\left(T_1 - T_0\right)}{1cm} = -k \cdot A_{cs} \frac{\left(T_2 - T_1\right)}{1cm} + h \cdot \left(\pi \cdot 1cm \cdot 1cm\right)\left(T_1 - T_{fluid}\right)$$

No energy balance is needed for element 0, because its temperature is known. For element 5, the energy balance is

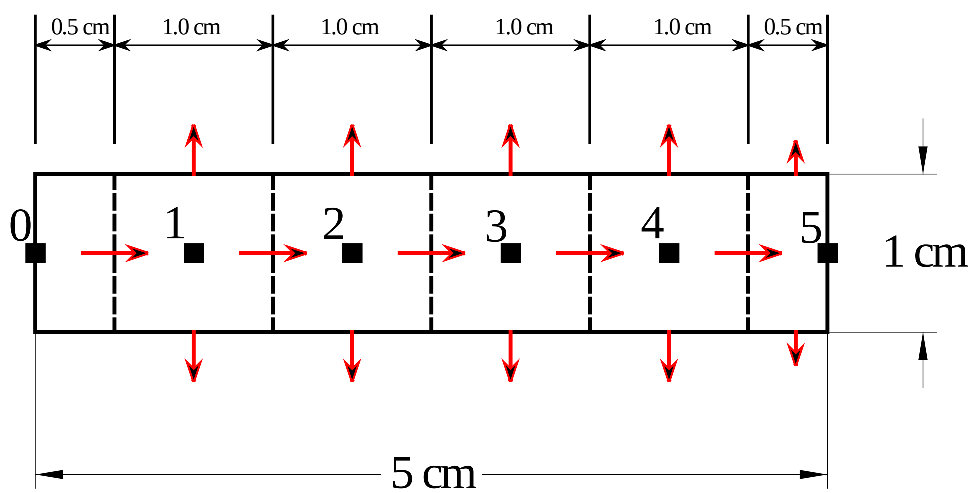

Figure 2: Subdivided Fin.

$$-k \cdot A_{cs} \frac{(T_5 - T_4)}{1cm} = +h \cdot (\pi \cdot 1cm \cdot 0.5cm)(T_5 - T_{fluid})$$

Note that the length of element 5 is 0.5 cm rather than 1.0 cm.

Ultimately, there are five energy balances, one for each element except element 0. There are also five unknown temperatures T_1 through T_5.

Solving the Equations

Mathcad software was used to solve the five equations for the five unknown temperatures. Because this is the students' first exposure to finite element analysis, the energy balance equations were left in the form shown above. It is possible, of course, to streamline these equations, and this is done in later applications, but the author feels that the analysis is less mysterious to students if the equations are not streamlined at first.

A portion of the Mathcad worksheet is shown in Figure 3 below:

The resulting temperature values T_1 through T_5 are shown in Figure 4 below. The discrete points in the graph come from the exact solution, while the line represents the finite element solution. At each discrete point, the two solutions differ by less than 1ºC.

Calculation of the rate of heat transfer is simply a matter of multiplying the convection coefficient times the surface area of each element times the difference between the uniform temperature of that element and the temperature of the fluid. In this case, the sum is 56.0 Watts. From the exact solution for insulated tip fins, the value is 55.8 Watts, a discrepancy of 0.36%.

Given

$$-k_{fin} \cdot A_{cs} \cdot \frac{T_1 - T_0}{\Delta x} = -k_{fin} \cdot A_{cs} \cdot \frac{T_2 - T_1}{\Delta x} + h \cdot (P_{fin} \cdot \Delta x) \cdot (T_1 - T_{fluid}) \quad \text{Element 1}$$

$$-k_{fin} \cdot A_{cs} \cdot \frac{T_2 - T_1}{\Delta x} = -k_{fin} \cdot A_{cs} \cdot \frac{T_3 - T_2}{\Delta x} + h \cdot (P_{fin} \cdot \Delta x) \cdot (T_2 - T_{fluid}) \quad \text{Element 2}$$

$$-k_{fin} \cdot A_{cs} \cdot \frac{T_3 - T_2}{\Delta x} = -k_{fin} \cdot A_{cs} \cdot \frac{T_4 - T_3}{\Delta x} + h \cdot (P_{fin} \cdot \Delta x) \cdot (T_3 - T_{fluid}) \quad \text{Element 3}$$

$$-k_{fin} \cdot A_{cs} \cdot \frac{T_4 - T_3}{\Delta x} = -k_{fin} \cdot A_{cs} \cdot \frac{T_5 - T_4}{\Delta x} + h \cdot (P_{fin} \cdot \Delta x) \cdot (T_4 - T_{fluid}) \quad \text{Element 4}$$

$$-k_{fin} \cdot A_{cs} \cdot \frac{T_5 - T_4}{\Delta x} = h \cdot \left(P_{fin} \cdot \frac{\Delta x}{2}\right) \cdot (T_5 - T_{fluid}) \quad \text{Element 5}$$

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} := \text{Find}(T_1, T_2, T_3, T_4, T_5)$$

Figure 3: Part of Mathcad Worksheet.

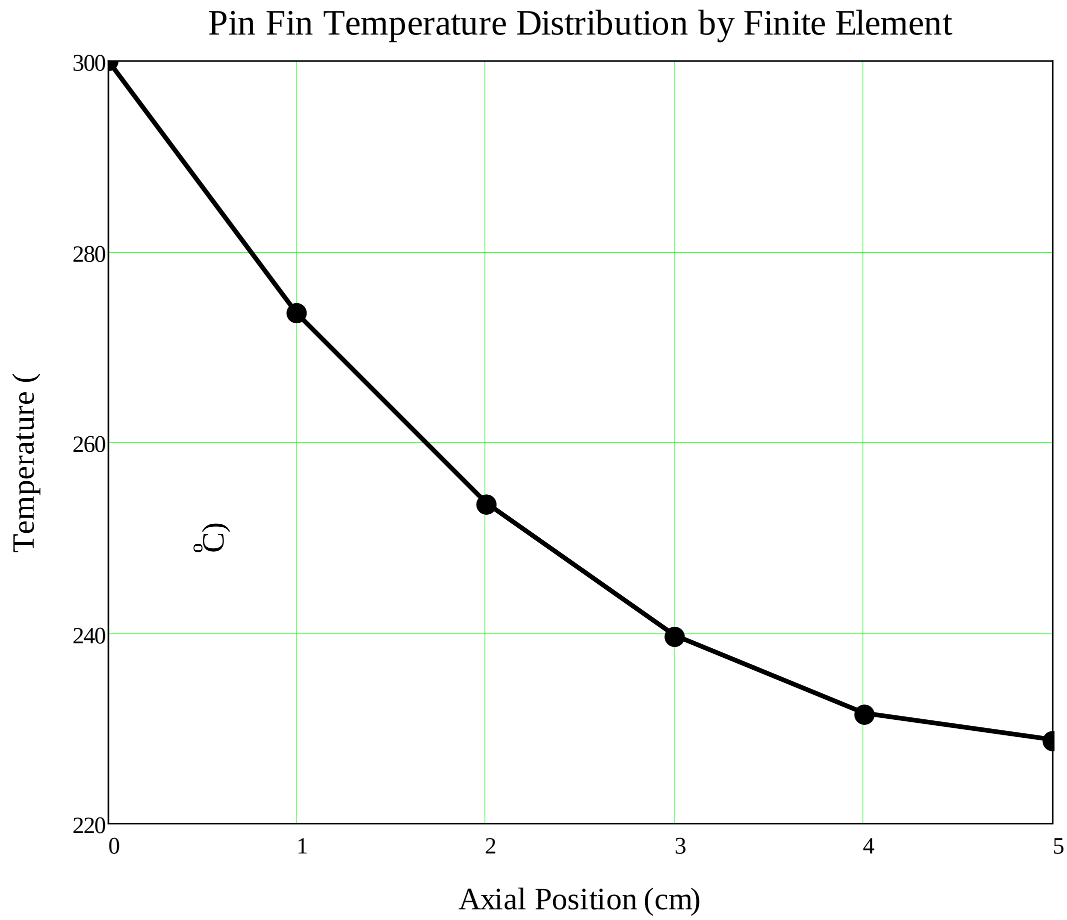

Figure 4: Comparison of Finite Element and Exact Solutions.

Extending the Analysis

A convection tip fin was easily modeled by changing the energy balance for element 5 to reflect convection from the tip. As with the insulated tip, the results from the finite element analysis are very close to the exact solution.

Modeling a conical fin was a bit more complicated, but easily understood. Elements 0 through 4 are frustums of cones, and element 5 is a small cone. A cross-section view of the conical fin is shown in Figure 5 below.

The area where two elements meet and conduction occurs varies with position along the fin. The surface area where convection occurs is

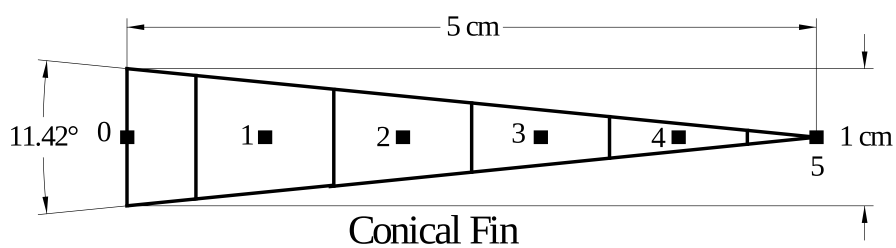

Figure 5.

the appropriate frustum surface area for elements 0 through 4 and the surface area of a cone for element 5. With these straightforward modifications to the Mathcad worksheet, the temperature distribution in a conical fin was calculated. Figure 6 below shows the temperature distributions in the three fin scenarios.

In the conical fin case, the author is not aware of an exact solution for the temperature distribution. There is, however, an exact solution for the rate of heat transfer from the fin. Its value is 30.7 Watts, while finite element analysis gives an estimate of 28.0 Watts. An examination of the nature of the FEA approximation suggests that its estimate is expected to be too low. In the FEA approximation each element is assumed to be at a uniform temperature, but actually the temperature is higher where the diameter of the element is larger and lower where the diameter is smaller. Therefore near the left (hotter) end of each element, the actual rate of heat transfer per unit area is greater, and the area per unit length is greater. Assumption of a uniform temperature throughout the element does not account for this effect. A more accurate approximation should result from the use of elements that are shorter in the axial direction.

Conclusions

Finite element analysis of heat transfer in a pin fin was easily implemented in Mathcad software. Even with a small number of finite elements, the results are in excellent agreement with the exact solution for an insulated tip cylindrical fin and a convection tip cylindrical fin. For the conical fin, there is a significant error with a well-understood explanation. In the Mathcad worksheet, the governing equations are left in physically recognizable form to facilitate student understanding.

This simple use of finite element analysis in a junior level course serves as an introduction to the concepts of FEA. In later courses, students are introduced to the use of commercial FEA codes, primarily for structural analysis.

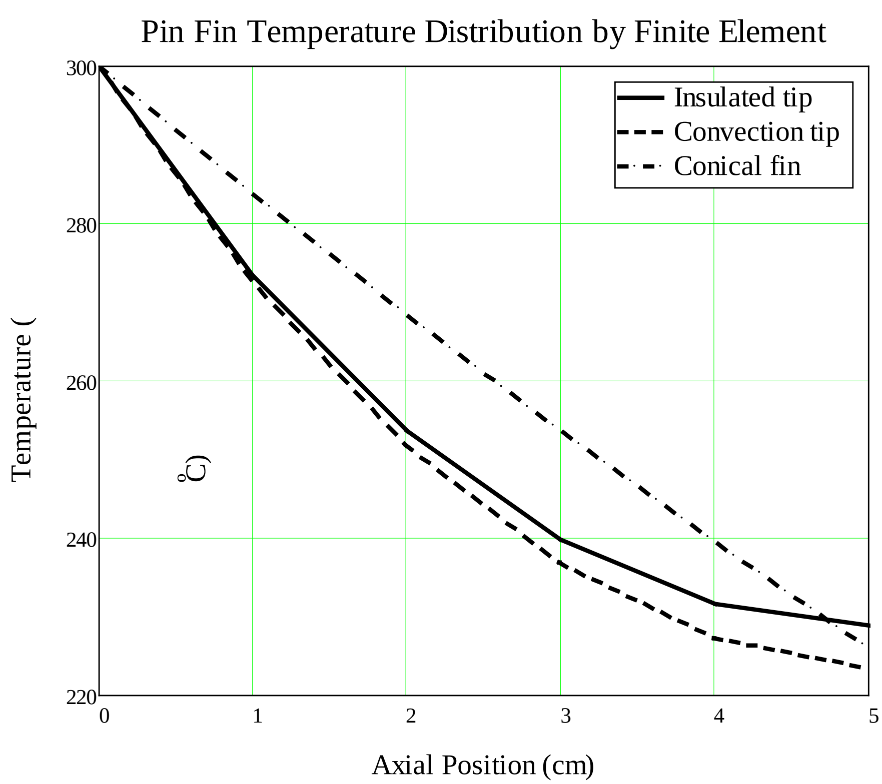

Figure 6: Comparison of Pin Fin Temperature Distributions.

Biographical Information

Edwin G. Wiggins holds B.S., M.S., and Ph.D. degrees in chemical, nuclear, and mechanical engineering, respectively, from Purdue University. He is the Mandell and Lester Rosenblatt Professor of Marine Engineering at Webb Institute in Glen Cove, NY. He is a past chairman of the New York Metropolitan Section of the Society of Naval Architects and Marine Engineers (SNAME) and a past regional vice president of SNAME. A Centennial Medallion and a Distinguished Service Award recognize his service to SNAME. As a representative of SNAME, he has served on the Engineering Accreditation Commission, the Technology Accreditation Commission, and the Board of Directors of the Accreditation Board for Engineering and Technology (ABET).