ROTATIONS AND QUATERNIONS IN 3- AND 4-DIMENSIONAL SPACES

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Abstract

We illustrate that applications of rotations and motions to computer graphics may be implemented by using quaternions. As an example, we analyze an interesting function published in a seminal paper in 1931 by Heinz Hopf. The function has many interesting geometric properties. The Hopf function provides a tiling of $\mathbb{R}^3$ using circles and one single line, as seen in Figures 1 and 2.

Introduction

Geometric tools are important in solving many engineering and computer graphic problems. Specially if the problems involve rigid motions. In 1931, Heinz Hopf published a paper concerning a very interesting function between two spheres, see [3]. We call this function the Hopf map. Using the language of algebraic topology briefly, the Hopf map is a generator of the homotopy group of continuous functions from S^3 to S^2. Before the discovery of the Hopf map it was not known whether this particular homotopy group contained only one homotopy class, as discussed in [6]. Now we know that the homotopy group is isomorphic to the group of integers $\mathbb{Z}$.

In this paper we describe interesting geometrical implications of Hopf's mapping, and apply Mathematica to draw objects in 3-dimensional space. For example, the function provides a way of tiling the 3-dimensional Euclidean space $\mathbb{R}^3$ by infinitely many circles and a single line each of which is linked to a fixed unit circle. We will use quaternions to describe Hopf's function. For more historical anecdotes, Richard Hamilton discovered the quaternions in 1843 and Coxeter used quaternions to characterize rotations in $\mathbb{R}^3$ and $\mathbb{R}^4$ in a paper published in 1946. We intend this paper to serve as a survey of results, and as such complicated mathematical proofs are omitted. However, we include some of the basic prerequisite ideas.

The Hopf Map

We briefly review the quaternions, see [1,2,5]. A quaternion x can be written in the form

$$x = a + bi + cj + dk$$

where $a, b, c, d \in \mathbb{R}$. The quaternions are added component-wise, e.g.,

$$(3i + 4j) + (-5j + k) = 3i - j + k.$$

To multiply two quaternions we assume the identities below:

1. $i^2 = j^2 = k^2 = ijk = -1$, and
2. $ij = -ji = k, jk = -kj = i, ki = -ik = j.$

We denote the set of quaternions by $\mathbb{H}$. The norm of x is defined as

$$N[x] = a^2 + b^2 + c^2 + d^2.$$

We say x is a unit quaternion if $N[x] = 1$.

If $a = 0$, we call x a pure quaternion. Consider the following subsets of $\mathbb{H}$:

$$S^3 = \{x \in \mathbb{H} : N[x] = 1\}$$

$$S^2 = \{x \in \mathbb{H} : N[x] = 1, \; x \text{ is a pure quaternion}\}$$

We may think of a quaternion x as a point $(a, b, c, d) \in \mathbb{R}^4$. Then we can realize $\mathbb{H}$ as $\mathbb{R}^4$. Consequently, S^3 and S^2 are naturally identified with the unit spheres in $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively.

The conjugate of x is $\bar{x} = a - bi - cj - dk$. Since $x\bar{x} = N[x]$, $x^{-1} = \bar{x}$ provided $N[x] = 1$. Unlike the complex numbers, conjugation is an anti-homomorphism, i.e. $\overline{x_1 x_2} = \overline{x_2}\,\overline{x_1}$ for $x_1, x_2 \in \mathbb{H}$ Also, we have $(x_1 x_2)^{-1} = {x_2}^{-1} {x_1}^{-1}$.

Using the quaternions the Hopf map is defined as follows:

$$h : S^3 \to S^2, \ h(x) = xkx^{-1}$$

The *fibers* of the Hopf map are

$$h^{-1}(y) = \{x \in S^3 \mid h(x) = y\}, y \in S^2.$$

In particular, the fiber over $y = k \in S^2$ can be shown to be

$$h^{-1}(k) = \{x \in S^3 \mid xk = kx\} = \{\sin\alpha + \quad k\cos\alpha \mid 0 \leq \alpha \leq 2\pi\}$$

The fiber $h^{-1}(k)$ is a circle in S^3 of radius 1 and centered at the origin. In fact, the Hopf map is surjective. For if $y \neq \pm k$, and

$$y = Ai + Bj + Ck \in S^2$$

and if we let

$$q_y = \sqrt{\frac{1+C}{2}} - B\sqrt{\frac{1-C}{2(A^2+B^2)}}\,i + A\sqrt{\frac{1-C}{2(A^2+B^2)}}\,j$$

then we can show that $q_y \in S^3$ and $h(q_y) = y$, however we leave out the details. Furthermore, the fiber of y can be shown to satisfy

$$h^{-1}(y) = q_y \cdot h^{-1}(k) = \{q_y \cdot x : x \in h^{-1}(k)\}.$$

We remark that the fiber $h^{-1}(y)$ is another circle in S^3 of radius 1 and centered at the origin. This is because the mapping $x \mapsto q_1 x q_2$ is a rotation in S^3 if q_1, q_2 are unit quaternions, as discussed by Coxeter in [1].

Applying a Stereographic Projection

In [5], we find a good geometric discussion of the fibers of the Hopf map. To help visualize images in S^3 we apply a stereographic projection that maps S^3 onto $\mathbb{R}^3 \cup \{\infty\}$. Given a point $x \in S^3$ with $x \neq k$ we consider the line through x and k. The line intersects the xyz-space or $\mathbb{R}^3$ at a certain point that we denote by $\Pi_3(x)$. This correspondence is given by a function $\Pi_3 : S^3 \to \mathbb{R}^3 \cup \{\infty\}$ where

$$\Pi_3(a + bi + cj + dk) = \left(\frac{a}{1-d}, \frac{b}{1-d}, \frac{c}{1-d}\right)$$

and $\Pi_3(k) = \infty$. For example, if we apply the projection to the fiber $h^{-1}(k)$ we obtain

$$\Pi_3(h^{-1}(k)) = \left\{\left(\frac{\sin\alpha}{1-\cos\alpha}, 0, 0\right) : \ 0 \leq \alpha \leq 2\pi\right\}$$

which is the x-axis in $\mathbb{R}^3$ union with $\{\infty\}$. Also, we find that the projection of the fiber $h^{-1}(-k)$ is the unit circle in the yz-plane, i.e.

$$\Pi_3\big(h^{-1}(-k)\big) = \{(0, \cos\alpha, \sin\alpha) : 0 \leq \alpha \leq 2\pi\}$$

Moreover, if $y \neq \pm k$ we claim that the projection of the fiber $h^{-1}(y)$ into $\mathbb{R}^3$ is a circle that passes through the interior and the exterior of the circle $y^2 + z^2 = 1$ in the yz-plane. Since we intend that this paper be geometric in nature we omit long computational proofs. And, the plane of the circle $\Pi_3\big(h^{-1}(y)\big)$ contains the origin. The fibers of the Hopf function provide a tiling of $\mathbb{R}^3$ since

$$\mathbb{R}^3 = \bigcup_{y \in S^2} \Pi_3\big(h^{-1}(y)\big)$$

Applying Mathematica we show below some of the fibers that tile $\mathbb{R}^3$.

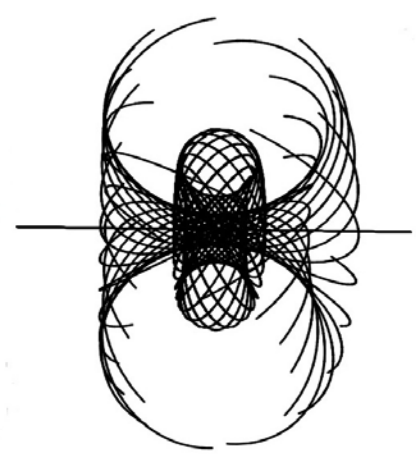

Figure 1: A tiling of $\mathbb{R}^3$ using circles and a line each of which is linked to one unit circle.

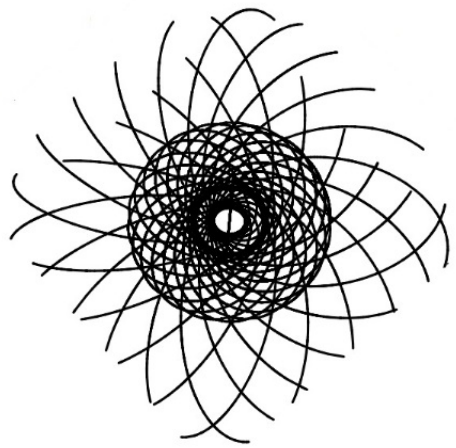

Figure 2: The same tiling of $\mathbb{R}^3$ from another point of view.

Student Research Projects

The above project was studied by the first author in an independent reading project. The second author served as the adviser of the project. Also, we analyzed the fibers under the Hopf map of other images in S^2 by projecting these fibers into $\mathbb{R}^3$ by using the stereographic projection Π_3. The quaternions have applications to motion control, robotics, and computer graphics, see [4.]

At Southeastern Louisiana University majors in mathematics must complete 43 semester hours in mathematics beginning with calculus. An independent project is not required of a math major but is ideally encouraged. The Math Department has designated courses with 1-3 units where students can be given credit for independent research. Students usually approach faculty members and discuss interests in research. In the past two years, a sample of other research topics included determinants of a class of matrices, extension fields and sums of orthogonal matrices, and sums of sines and cosines. These latter projects were supervised by Professor Dennis Merino.

Also, our University offers an interdisciplinary graduate program ISAT with emphasis in chemistry, computer science, industrial technology, mathematics, and physics. The program requires 33 semester hours which includes an independent research project or thesis.

Finally, we thank Professor Carl Steidely for his comments on the engineering applications, and to Professor Gary Walls for valuable editorial suggestions.

References

1. Coxeter, H.S.M., “*Quaternions and Reflections*", American Mathematical Monthly, 53 (1946),136-146.

2. Gallier, Jean, *Geometric Methods and Applications*, Texts in Applied Mathematics, Vol. 38, 2001.

3. Hopf, Heinz, *Uber die Abbildungen der dreidimensionalen Sphare auf die Kugelflache*, Math. Ann. 104 (1931), No. 1, 637-665.

4. Kuipers, J.B., *Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace and Virtual Reality*, Princeton University Press, 2002.

5. Lyons, David, *An Elementary Introduction to the Hopf Fibration*, Mathematics Magazine, Vol. 76, No. 2 (April 2003), 87-98

6. James, I.M., *History of Topology*, North-Holland, 1999, 575-578

Biographical Information

Recep Avci graduated with a master's degree in Integrated Science and Technology at Southeastern Louisiana University in August 2010. He continues his higher education in Applied Computing program at the University of Central Arkansas. His research interests are computational modeling and image processing in Bioinformatics.

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