

SOLVING VIBRATION ANALYSIS DIFFERENTIAL EQUATIONS BY NUMERICAL TECHNIQUES

Cyrus K. Hagigat
College of Engineering
The University of Toledo

Introduction

Motions of vibrating systems are governed by differential equations. The differential equation for a single degree of freedom system consists of mass, stiffness, damping, mass displacement, mass velocity and mass acceleration for the system. There always is an exact solution available for a single degree of freedom differential equation.

Motions for multi degree of freedom systems are described by individual differential equations for each degree of freedom. The individual differential equations have elements of other degrees of freedom in them. Consequently, no closed form solution for systems that have 3 or more degrees of freedom and have damping terms is available.

The theme of this article is developing a numerical method and programming it in EXCEL and verifying the validity of the numerical solution by comparing it against the known exact solution.

The numerical technique is then expanded to two degree of freedom vibrating systems, and the methods programmed in EXCEL.

Examples of vibration scenarios for both single degree of freedom and two degree of freedom vibrating systems are calculated using EXCEL and plotted for illustration.

The techniques presented in this article are used in an advanced engineering mathematics course to illustrate the physical meanings behind calculus (integration, differentiation & differential equations), and illustrating computerized techniques for solving equations that have no exact solutions.

Nomenclature

C: Damping of a single degree of freedom vibrating system (lb-sec/inch).

F_i : Force applied to a mass at time increment i .

h : time interval for numerical solution (sec).

K: Stiffness of a single degree of freedom vibrating system (lb/inch).

m : Mass of a single degree of freedom vibrating system (lb-sec²/inch).

X: displacement for mass of a single degree of freedom vibrating system (inch).

dX/dt : velocity for mass of a single degree of freedom vibrating system (inch/sec).

d^2X / dt^2 : acceleration for mass of a single degree of freedom system (inch²/sec).

m_1 : First mass of a two degree of freedom vibrating system (lb-sec²/inch).

X_1 : displacement for mass 1 of a two degree of freedom vibrating system (inch).

dX_1/dt : velocity for mass 1 of a two degree of freedom vibrating system (inch/sec).

d^2X_1 / dt^2 : acceleration for mass 1 of a single degree of freedom system (inch²/sec).

m_2 : Second mass of a two degree of freedom vibrating system (lb-sec²/inch).

X_2 : displacement for mass 2 of a two degree of freedom vibrating system (inch).

dX_2/dt : velocity for mass 2 of a two degree of freedom vibrating system (inch/sec).

d^2X_2 / dt^2 : acceleration for mass 2 of a single degree of freedom system (inch²/sec).

$X_i, (X1)_i, (X2)_i =$ Displacement at increment i (inch) ($i=1, 2, 3, \dots$)

$(dX / dt)_i, (dX1 / dt)_i, (dX2 / dt)_i =$ velocity at increment i (inch/sec) ($i=1, 2, 3, \dots$)

$(d^2X / dt^2)_i, (d^2X1 / dt^2)_i, (d^2X2 / dt^2)_i =$ Acceleration at increment i (inch/ sec²) ($i=1, 2, 3, \dots$)

$\omega_n =$ Natural frequency of a free un-damped vibrating system.

Technical Discussion

Figure 1 is a graphical representation for a single degree of freedom vibrating system.

The equation of a single degree of freedom vibrating system is shown in equation. [1]

$$m d^2X / dt^2 + C dX / dt + KX = F \quad (1)$$

Based on a Taylor series approximation, equations (2) & (3) can be obtained for finding

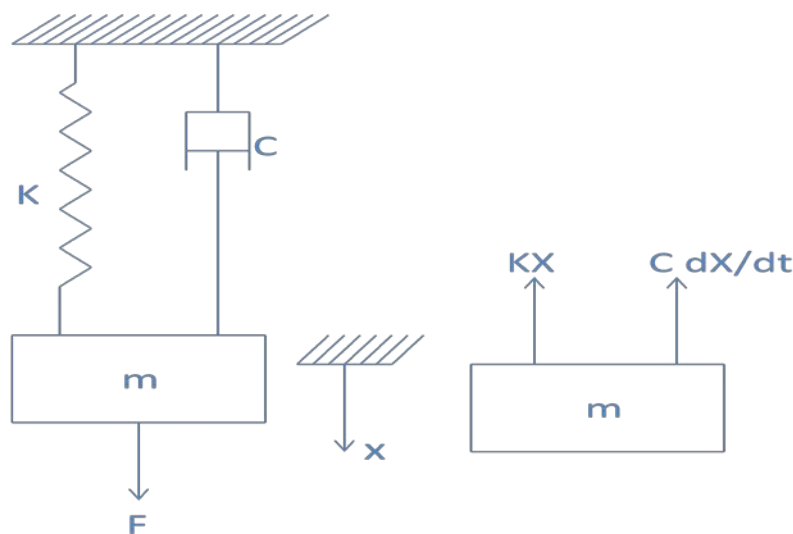


Figure 1: Graphical representation of a single degree of freedom vibrating system.

first and second order derivatives of a function X with respect to time. In equations (2) & (3), h is the time interval. For the approximation to produce acceptable results, h must be small enough for the results to be close to exact solutions, but not too small such that excessive numerical errors are accumulated as a result of too many solution steps. [2], [4]

$$X_{i+1} = X_i + h (dX / dt)_i + \{(h^2 / 2) (d^2X / d t^2)_i\} \quad (2)$$

$$X_{i-1} = X_i - h (dX / dt)_i + \{(h^2 / 2) (d^2X / d t^2)_i\} \quad (3)$$

Adding equations (2) & (3) results in equation (4).

$$X_{i+1} = 2 X_i - X_{i-1} + \{h^2 (d^2X / d t^2)_i\} \quad (4)$$

Rearranging equation (3) results in equation (5).

$$(dX / dt)_i = -(X_{i-1} / h) + (X_i / h) + \{(h / 2) (d^2X / d t^2)_i\} \quad (5)$$

The following is the derivation of equations that can be used for a numerical solution of a 1 degree of freedom system assuming the initial conditions of applied force, displacement, velocity and acceleration are known.

Equation (2) with $i=0$ results in:

$$X_1 = X_0 + h (dX / dt)_0 + \{(h^2 / 2) (d^2X / d t^2)_0\} \quad (6)$$

Equation (5) with $i=1$ results in:

$$(dX / dt)_1 = -(X_0 / h) + (X_1 / h) + \{(h / 2) (d^2X / d t^2)_1\} = -(X_0 / h) + (X_1 / h) + (h/2)\{f [X_1, (dX / dt)_1]\} \quad (7)$$

Combining equations (7) & (1) eliminates $f [X_1, (dX / dt)_1]$ from equation (7), and the form shown in equation (8) is obtained for $(dx/dt)_1$.

$$(dX/dt)_1 = (-2 m X_0 + 2 m X_1 + F_0 h^2 - k h^2 X_1) / (2 m h + C h^2) \quad (8)$$

By using equation (1), the following expression for $(d^2X/ dt^2)_1$ is obtained.

$$(d^2X/ dt^2)_1 = \{F_0 - C (dX/dt)_1 - k X_1\} / m \quad (9)$$

Equation (4) with $i=1$ results in:

$$X_2 = 2 X_1 - X_0 + h^2 (d^2X / d t^2)_1 \quad (10)$$

Equation (5) with $i=2$ results in:

$$(dX / dt)_2 = -(X_1 / h) + (X_2 / h) + \{(h / 2) (d^2X / d t^2)_2\} = -(X_1 / h) + (X_2 / h) + (h/2)\{f [X_2, (dX / dt)_2]\} \quad (11)$$

Equation (11) can be modified by the same technique used to modify equation (7) to obtain the following expression for $(dX/dt)_2$.

$$(dX/dt)_2 = (-2 m X_1 + 2 m X_2 + F_1 h^2 - k h^2 X_2) / (2 m h + C h^2) \quad (12)$$

By similarity to equation (9) the following can be obtained.

$$(d^2X/ dt^2)_2 = \{F_1 - C (dX/dt)_2 - k X_2\} / m \quad (13)$$

By analogy and using equations (10), (12) and (13), equations (14), (15) and (16) are obtained for calculating displacement, velocity and acceleration for time step 3 of the numerical solution.

$$X_3 = 2 X_2 - X_1 + h^2 (d^2X / d t^2)_2 \quad (14)$$

$$(dX/dt)_3 = (-2 m X_2 + 2 m X_3 + F_2 h^2 - k h^2 X_3) / (2 m h + C h^2) \quad (15)$$

$$(d^2X/ dt^2)_3 = \{F_2 - C (dX/dt)_3 - k X_3\} / m \quad (16)$$

Calculations for additional time increments can be performed by incrementing the calculation indexes in equations (14), (15) and (16).

Assuming the damping term and forcing function for the system shown in figure 1 are zero (free vibrating system), the system would respond to an initial disturbance by vibrating at a frequency equal to the natural frequency of the system indefinitely (or until another force acts on the system). [1]

The natural frequency of the free vibrating system is defined by equation (17). [1]

$$\omega_n = (K/m)^{0.5} \quad (17)$$

The displacement of the free vibrating system is defined by equation (18). [1]

$$X = [\{(dX/dt)_0 / \omega_n\} \text{SIN} (\omega_n t)] + X_0 \text{COS} (\omega_n t) \quad (18)$$

Developing and Checking the Validity of the Numerical Solutions

Equations 6, 8, 9, 10, 12, 13, 14, 15 and 16 are used to start the numerical solution scheme in an EXCEL spread sheet, and then equations 13, 14 & 15 are used repeatedly by incrementing "i" in order to extend the solution in time domain for as long as desired.

To check the accuracy of the formulas and the EXCEL programming of the formulas, the results are checked against known characteristics of vibrating systems.

Equations (17) & (18) define the behavior of the vibrating system of figure (1) when $C=0$ & $F=0$. Equations (17) & (18) can be readily

programmed in an EXCEL spread sheet. Figure 2 is a plot of numbers calculated by the exact solution of equations (17) & (18).

The numerical scheme developed can be used to calculate the displacement of the one degree

of freedom system used to get the displacement curve of Figure 2. Figure 3 is such a figure.

A comparison of Figures 2 & 3 shows that the numerical formulation is accurate.

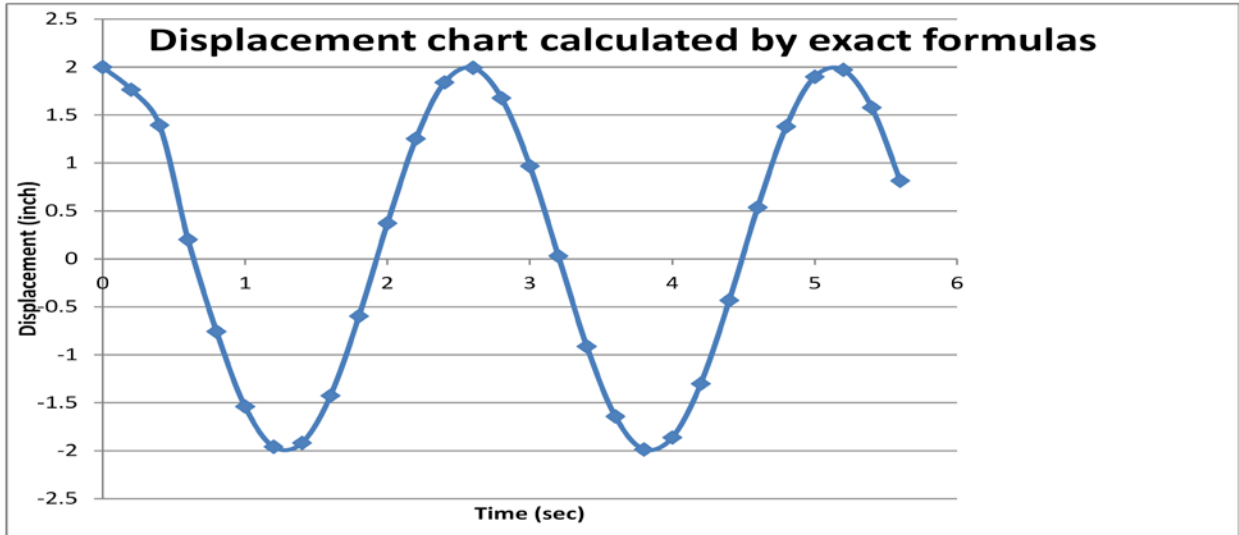


Figure 2: Displacement of vibration system of Figure 1 with $F=0$, $C=0$, $m=.5 \text{ Lb}\cdot\text{sec}^2/\text{inch}$, $K= 3 \text{ Lb}/\text{inch}$ and initial displacement of 2 inch calculated by an exact solution.

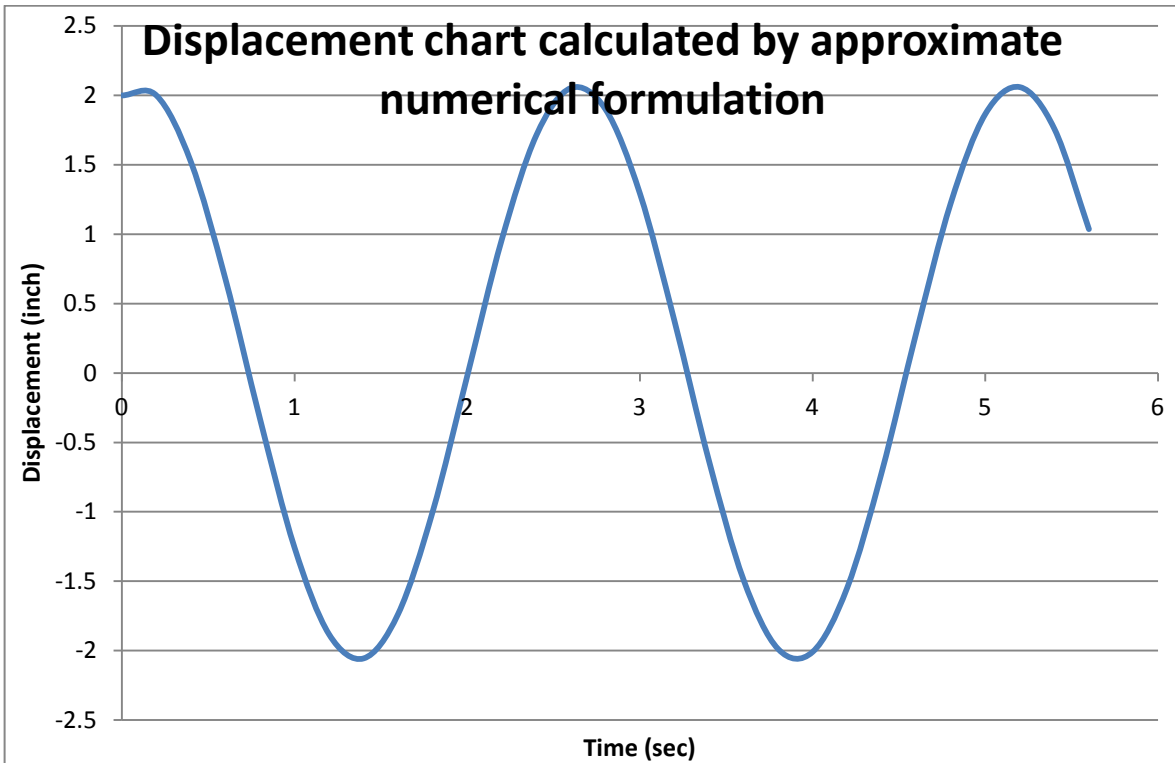


Figure 3: Displacement of vibration system of figure 1 with $F=0$, $C=0$, $m=.5 \text{ Lb}\cdot\text{sec}^2/\text{inch}$, $K= 3 \text{ Lb}/\text{inch}$, initial displacement of 2 inch, and $h=.2$ calculated by a numerical approximate solution.

Examples of Numerical Solutions of Vibration of a Single Degree of Freedom System

The numerical formulation can be programmed into an EXCEL spread sheet and different scenarios of a single degree of freedom vibrating system studied.

Scenario 1:

The following values are used as parameters of the vibrating system.

$$m = .5 \text{ lb. sec}^2/\text{inch}$$

$$c = .1 \text{ lb. sec/inch}$$

$$k = 3 \text{ lb/inch}$$

$$h = .2 \text{ sec}$$

$$F = 0$$

$$\text{Initial displacement} = 2 \text{ inch}$$

$$\text{Initial velocity} = 0$$

$$\text{Initial acceleration} = 0$$

Figure 4 is the calculation for the mass displacement. A review of Figure 4, and comparing Figure 4 to Figure 3 indicate that the vibration amplitude is reduced as a result of damping present in the calculations.

Scenario 2:

In certain vibrating scenarios, external conditions lead to self perpetuating vibration. This condition can be mathematically approximated by having a negative damping condition. For scenario 2 all the parameters remain the same except that $C = -0.1$

Figure 5 is a plot of displacements calculated by scenario 2. A comparison of Figures 4 & 5 confirm the expected result that while a positive damping reduces the vibration amplitude, a negative one will increase the vibration amplitude.

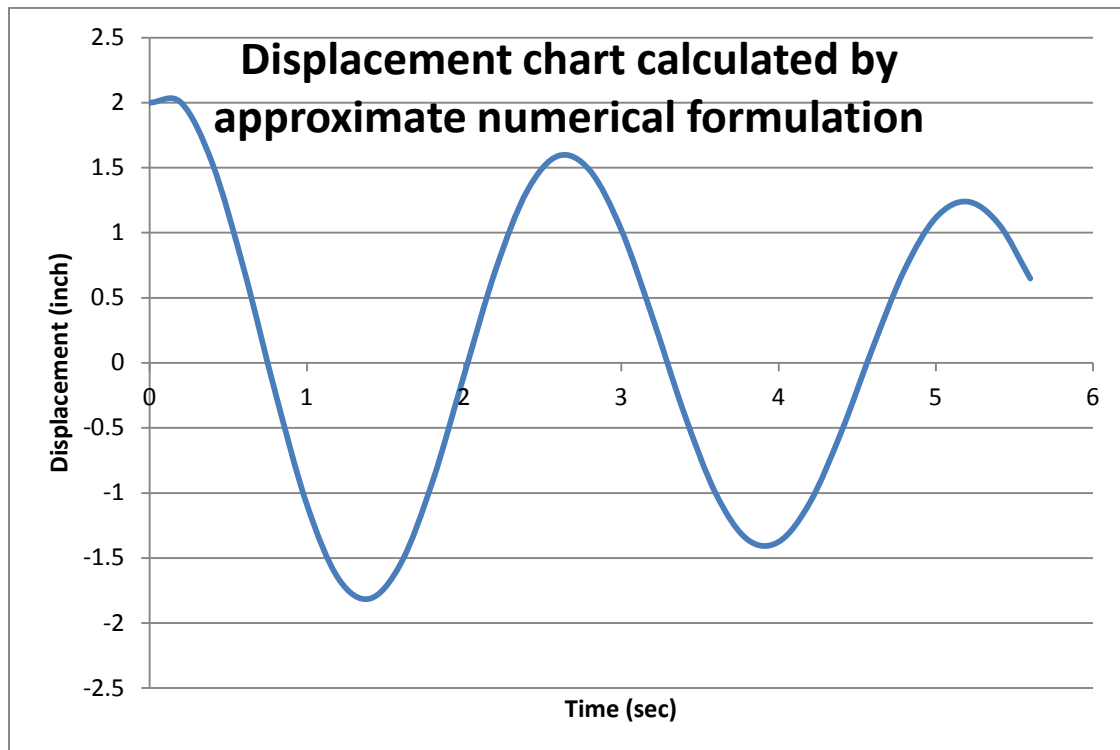


Figure 4: Results of vibration analysis of scenario 1.

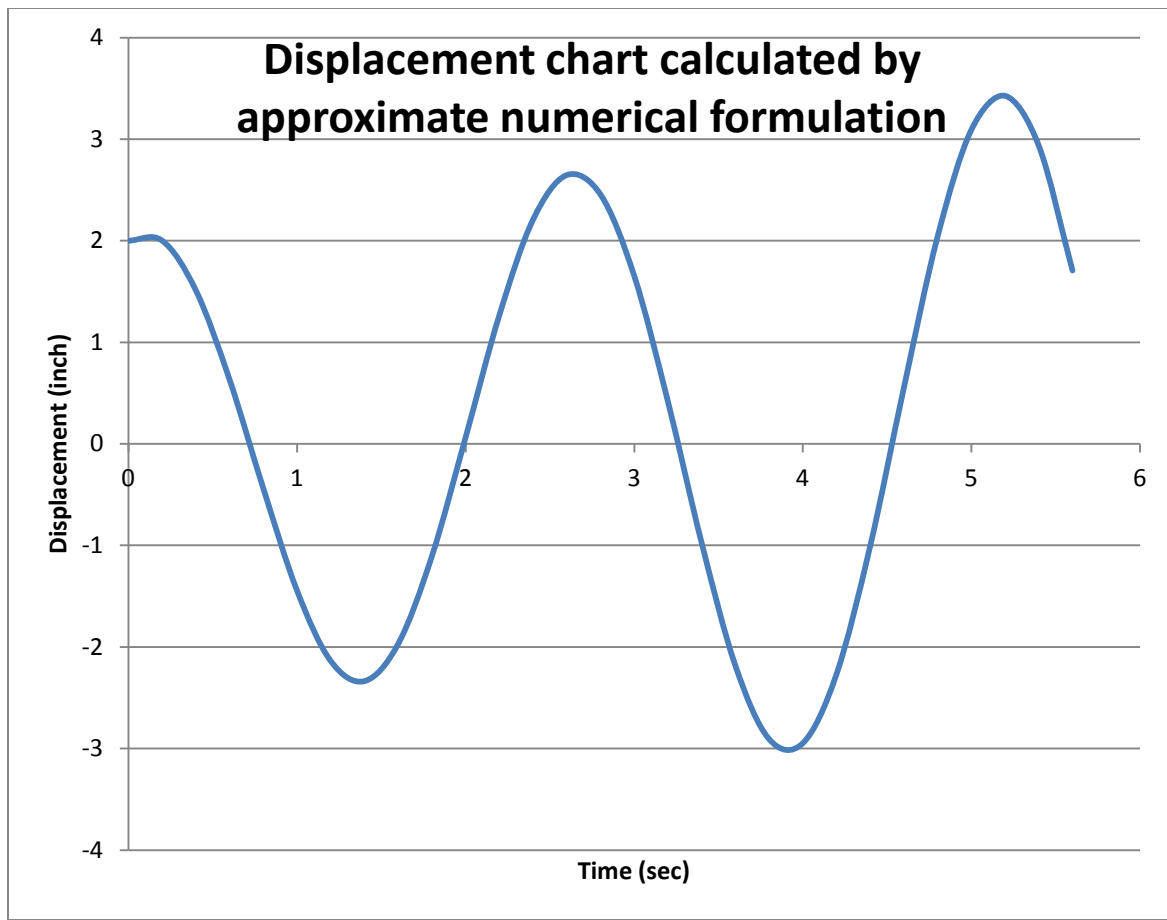


Figure 5: Results of vibration analysis of scenario 2.

Scenario 3:

It is known that if a vibrating system is excited at or near its natural frequency, the vibration amplitude will increase quickly.

Equation (17) can be used to calculate the natural frequency of a single degree of freedom system. Using equation (17) with the system characteristics of scenario 1, gives a natural frequency of 2.5 radians/sec.

Scenario 3 is the same as scenario 1 except that a sinusoidal force of “ $2 \sin(2.5t)$ ” is applied to the system, and the initial displacement is set to 0. Figure 6 is the result of the calculation for this scenario, a comparison of Figures 6 and 3 confirm the expected results.

Scenario 4:

Scenario 4 is the same as scenario 3, except that the damping coefficient is set to .3 lb.sec/inch instead of 0. A comparison of Figures 6 and 7 shows that the presence of damping suppresses the growth of vibration amplitude.

Scenario 5: As discussed in scenario 2, certain vibration scenarios can be modeled by a negative damping number. Scenario 5 is the same as scenario 4, except that the damping is -.3.

A comparison of Figures 6, 7 and 8 show that combining a negative damping with an external excitation at or close to the natural frequency of the system will lead to even a higher growth rate in vibration amplitude in comparison to positive damping or no damping.

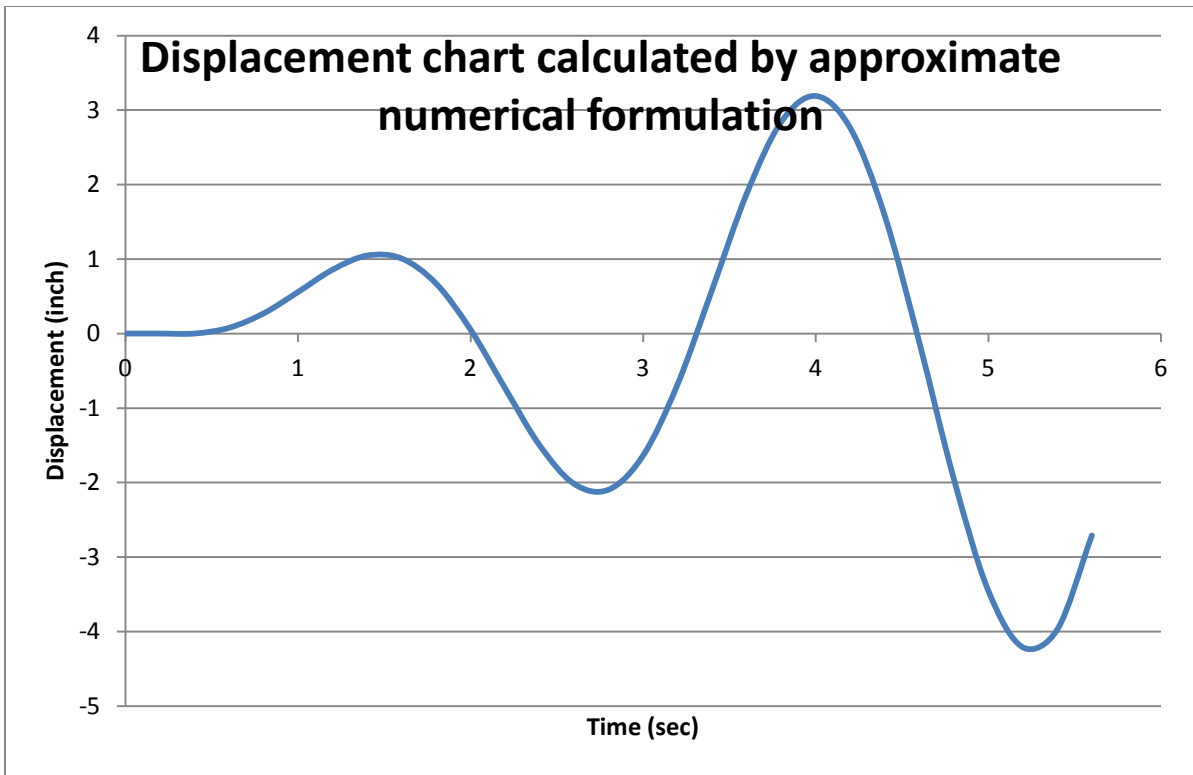


Figure 6: Results of vibration analysis of scenario 3.

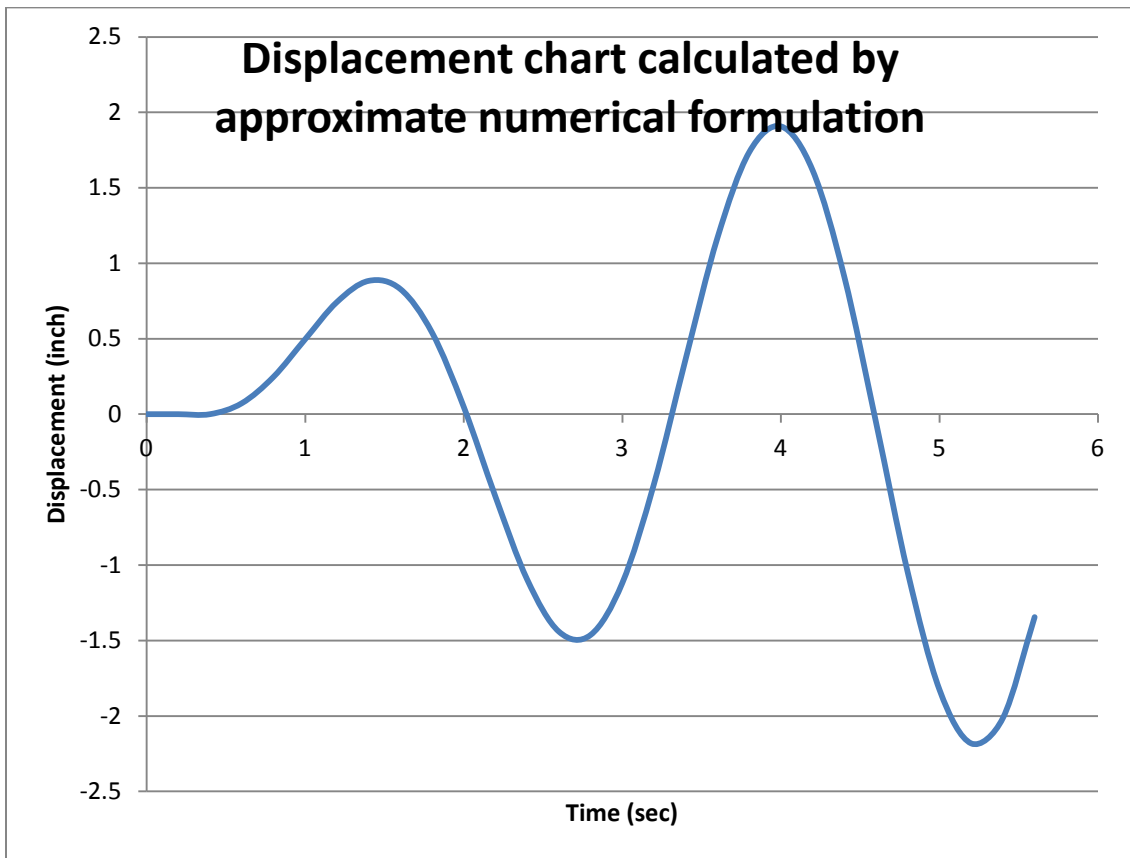


Figure 7: Results of vibration analysis of scenario 4.

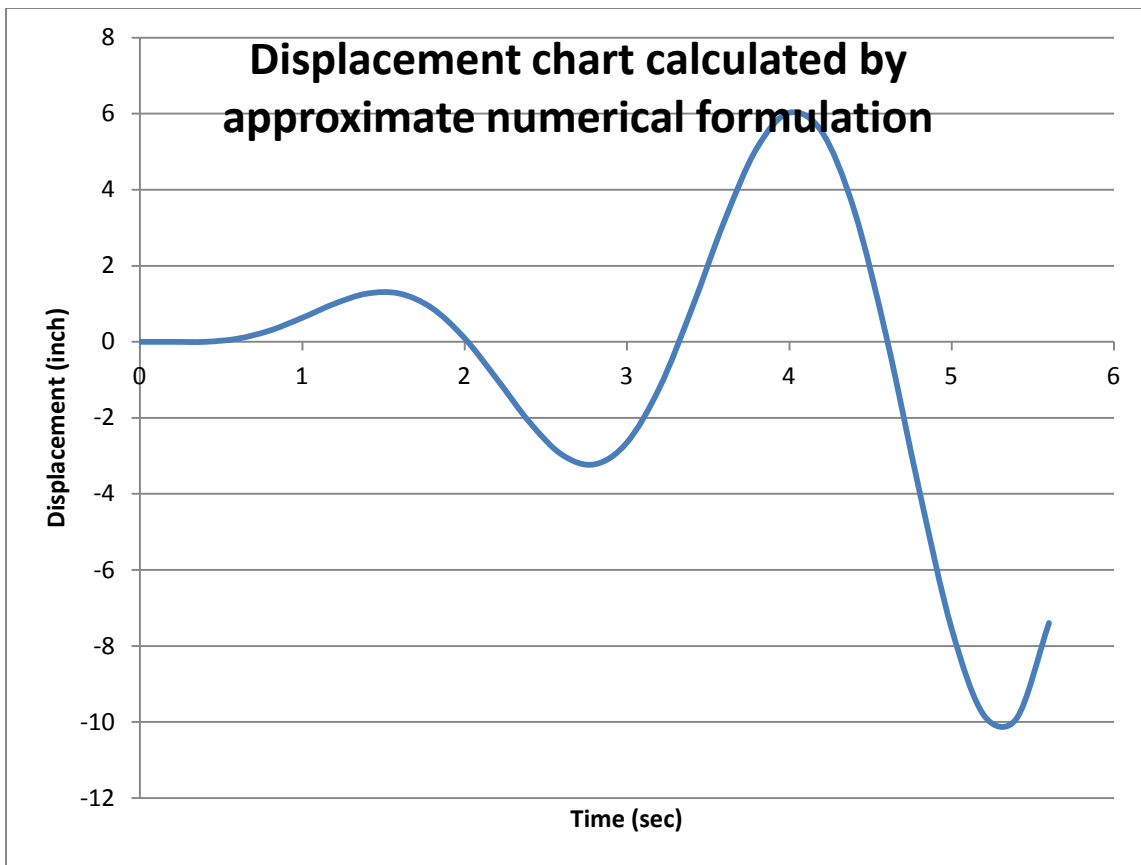


Figure 8: Results of vibration analysis of scenario 5.

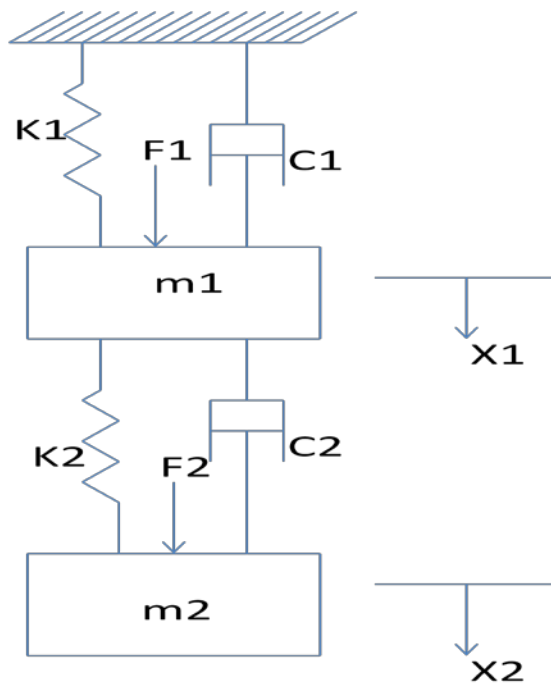


Figure 9: Graphical representation of a 2 degree of freedom vibrating system.

Extending the Numerical Solution to a 2 Degree of Freedom Vibrating System

Figure 9 is a graphical representation of a 2 degree of freedom vibrating system.

Equations (19) & (20) are the differential equations defining the motions of m_1 & m_2 . [3]

$$m_1. (d^2X_1/ dt^2) = -K_1.X_1 -K_2. (X_1 - X_2) - C_1. (dx_1/ dt) - C_1. (dX_1/ dt - dX_2/ dt) + F_1 \quad (19)$$

$$m_2. (d^2X_2/ dt^2) = -K_2. (X_2 - X_1) - C_2. (dX_2/ dt - dX_1/ dt) + F_2 \quad (20)$$

Equations (19) & (20) must be rearranged such that they are put in the same format as equation (1) so that a numerical scheme can be developed. Equation (21) is the rearranged version of equation (19) and equation (22) is the rearranged version of equation (20).

$$m_1. (d^2X_1/ dt^2) + C_1. (dX_1/ dt) + K_1.X_1 = F_1 -K_2. (X_1 - X_2) - C_1. (dX_1/ dt - dX_2/ dt) \quad (21)$$

$$m_2. (d^2X_2/ dt^2) + C_2. (dX_2/ dt) + K_2. X_2 = F_2 + C_2. (dX_1/ dt) + K_2.X_1 \quad (22)$$

Equations (21) & (22) are similar to a single degree of freedom equation except that the forcing functions on the right hand side of the equations are being manipulated by the relative motion of the masses. By remembering that the forcing functions are being manipulated and by using equations (21) & (22) and by analogy with the numerical scheme developed for a "1 degree of freedom" system, the following sequence of equations (equations 23 through 38) can be obtained for solving the system of figure 9 numerically. Equations (31) through (38) can be used repeatedly by increasing the subscripts in order to obtain a numerical solution for the system performance.

$$(G1)_0 = (F1)_0 - K_2 \{(X1)_0 - (X2)_0\} - C1 \{(dX1/dt)_0 - (dX2/dt)_0\} \quad (23)$$

$$(X1)_1 = (X1)_0 + h (dX1/dt)_0 + \{(h^2/2) (d^2X1/dt^2)_0\} \quad (24)$$

$$(dX1/dt)_1 = (-2 m_1 (X1)_0 + 2 m_1 (X1)_1 + (G1)_0 h^2 - k_1 h^2 (X1)_1) / (2 m_1 h + C_1 h^2) \quad (25)$$

$$(d^2X1/ dt^2)_1 = \{(G1)_0 - C_1 (dX1/dt)_1 - k_1 (X1)_1\} / m_1 \quad (26)$$

$$(G2)_0 = (F2)_0 + C_2 (dX1/dt)_0 + K_2 (X1)_0 \quad (27)$$

$$(X2)_1 = (X2)_0 + h (Dx2/dt)_0 + \{(h^2/2) (d^2X2/dt^2)_0\} \quad (28)$$

$$(dX2/dt)_1 = (-2 m_2 (X2)_0 + 2 m_2 (X2)_1 + (G2)_0 h^2 - k_2 h^2 (X2)_1) / (2 m_2 h + C_2 h^2) \quad (29)$$

$$(d^2X2/ dt^2)_1 = \{(G2)_0 - C_2 (dX2/dt)_1 - k_2 (X2)_1\} / m_2 \quad (30)$$

$$(G1)_1 = (F1)_1 - K_2 \{(X1)_1 - (X2)_1\} - C1 \{(dX1/dt)_1 - (dX2/dt)_1\} \quad (31)$$

$$(X1)_2 = 2(X1)_1 - (X1)_0 + \{h^2(d^2X1/dt^2)_1\} \quad (32)$$

$$(dX1/dt)_2 = (-2 m_1 (X1)_1 + 2 m_1 (X1)_2 + (G1)_1 h^2 - k_1 h^2 (X1)_2) / (2 m_1 h + C_1 h^2) \quad (33)$$

$$(d^2X1/ dt^2)_2 = \{(G1)_1 - C_1 (dX1/dt)_2 - k_1 (X1)_2\} / m_1 \quad (34)$$

$$(G2)_1 = (F2)_1 + C_2 (dX1/dt)_1 + K_2 (X1)_1 \quad (35)$$

$$(X2)_2 = 2 (X2)_1 - (X2)_0 \{h^2(d^2X2/dt^2)_1\} \quad (36)$$

$$(dX2/dt)_2 = (-2 m_2 (X2)_1 + 2 m_2 (X2)_2 + (G2)_1 h^2 - k_2 h^2 (X2)_2) / (2 m_2 h + C_2 h^2) \quad (37)$$

$$(d^2X2/ dt^2)_2 = \{(G2)_1 - C_2 (dX2/dt)_2 - k_2 (X2)_2\} / m_2 \quad (38)$$

Examples of Numerical Solutions of Vibration of a Two Degree of Freedom System

$K_1 = 3$ (lb/inch)
 $K_2 = 2$ (lb/inch)
 $C_1 = 1$ (lb-sec/inch)
 $C_2 = 2$ (lb-sec/inch)

Scenario 6:

This scenario involves applying an impulse force of 1 lb to m_2 of Figure 9. The following values are used in this scenario.

$m_1 = 0.5$ (lb-sec²/inch)
 $m_2 = 1$ (lb-sec²/inch)

Figure 10 is a plot of displacements for m_1 & m_2 of Figure 9.

Scenario 7:

This scenario is the same as scenario 6, except that the 1 lb impulse force is applied to m_1 .

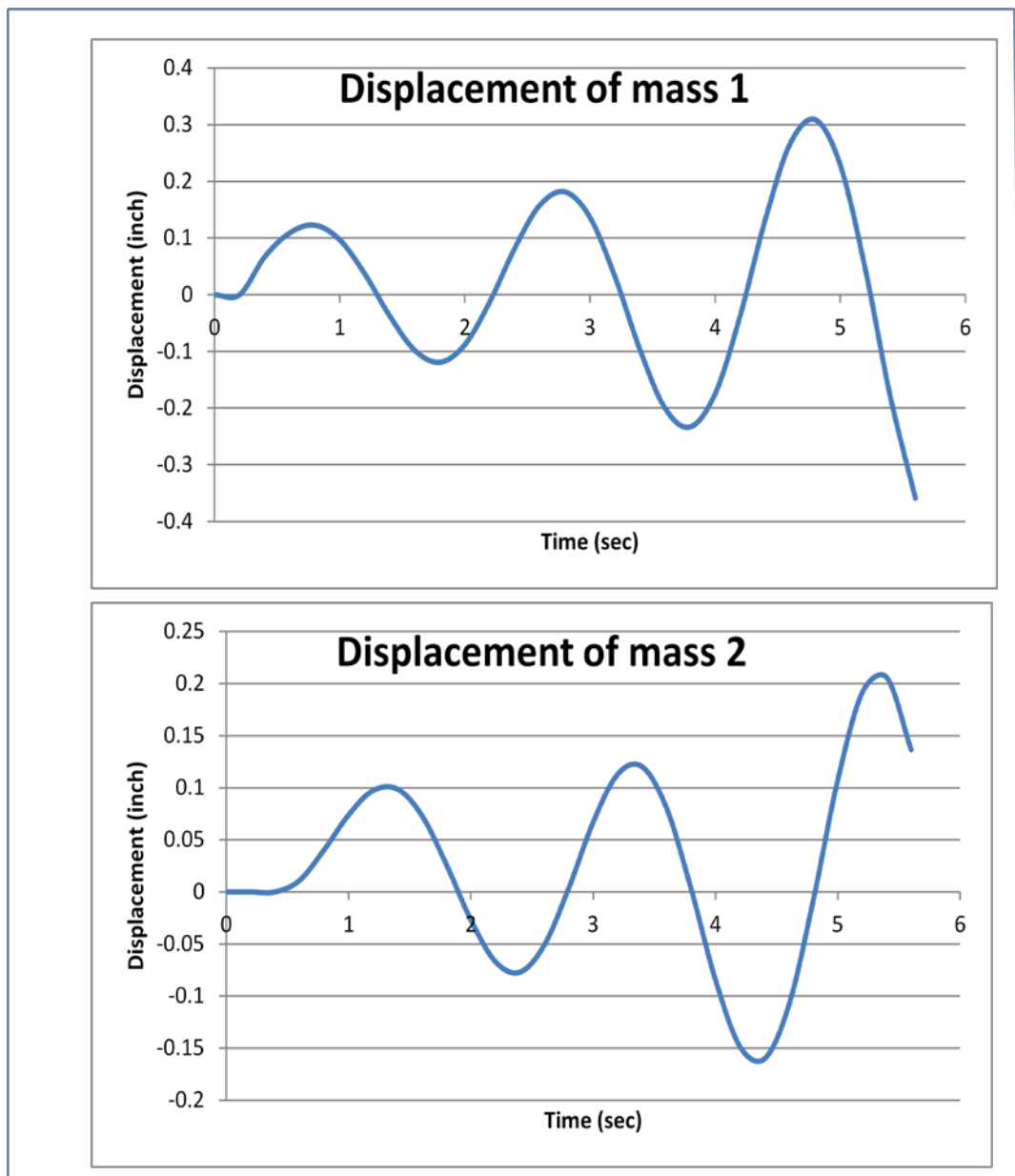


Figure 10: Displacements for m_1 & m_2 of figure 9 for scenario 6.

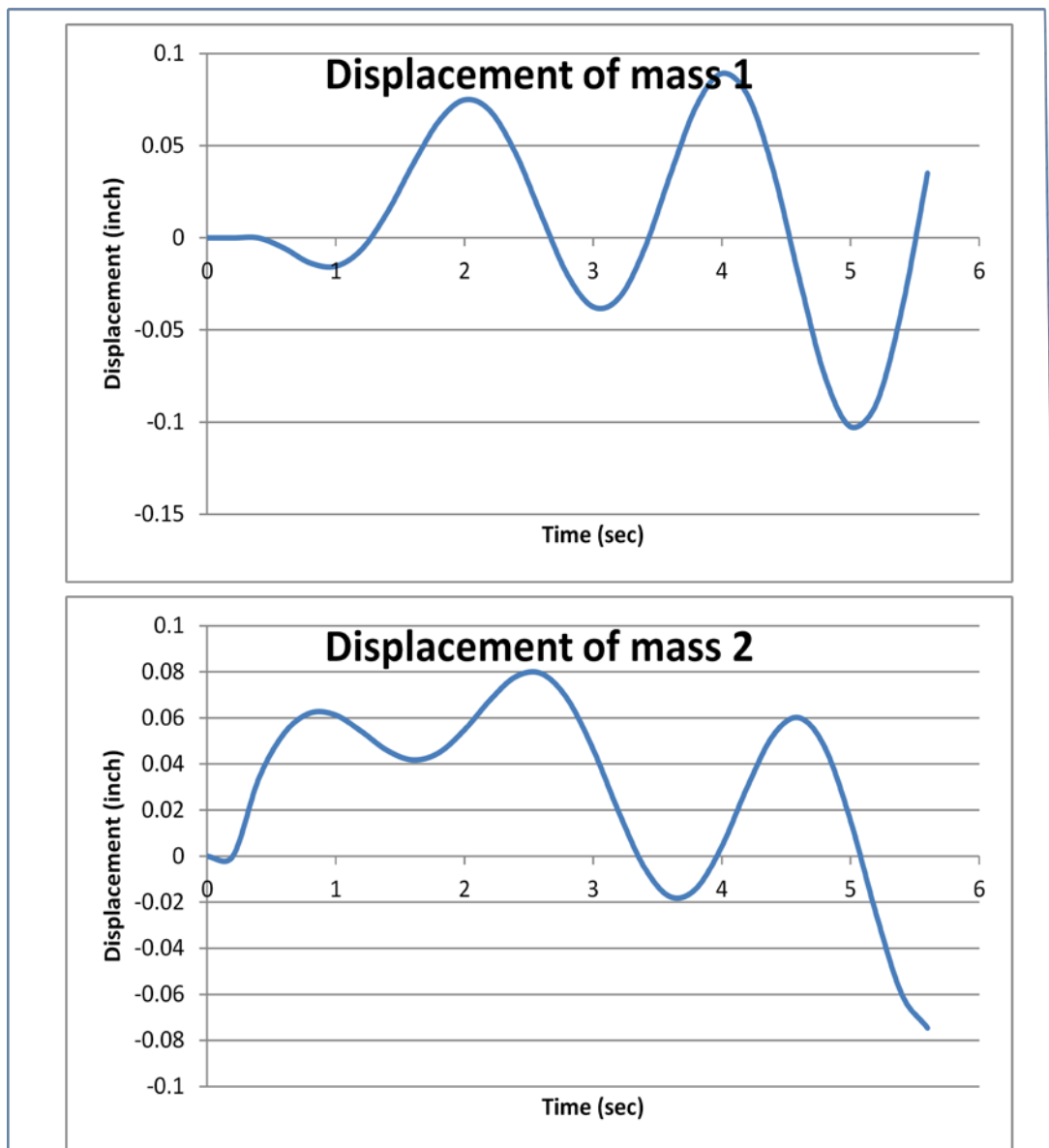


Figure 11: Displacements for m_1 & m_2 of figure 9 for scenario 7.

Pedagogical Component of Implementation

The concepts described in this article were used in an advanced engineering mathematics course in an engineering technology program. The emphasis in engineering technology courses is solving practical engineering problems compared to traditional engineering programs where emphasis is on theory.

The course is a required course for mechanical engineering technology (MET) and electronics engineering technology (EET) students. In fall 2012 semester 21 students started the class and

20 completed the course (1 student withdrew). The grades for fall of 2012 semester were from A through D-. In fall of 2011 semester 13 students started and completed the course. The grades for the fall of 2011 semester were A through C.

During fall of 2011 semester an electrical engineering problem (setting up and numerically solving a circuit scenario) was used. During fall of 2012 semester a mechanical engineering problem (vibration problem described in this article) was used. MET students had difficulty with the circuit project

and EET students had difficulty with the vibration project. In the future the author will assign a circuit based project to EET students and a vibration or fluid mechanics project to MET students, and teach the solution to both problems.

Overall, the students' understanding of why calculus and differential equation are needed in engineering was improved by doing the numerical projects. By doing the projects, the students developed a better understanding of why algebra is not enough for solving engineering problems. The concepts presented in this article give physical meaning to various calculus concepts. For example, it is shown that the derivative of a displacement term is velocity, and the derivative of velocity is acceleration. Then displacement, velocity and accelerations are all used in a vibrating system to give a physical meaning to each of them.

The concepts in this paper also demonstrate that not all differential equations have closed form solutions, and how numerical computer based solution techniques can be used where no known closed form solution exists. Solving differential equations by numerical techniques is a part of the course.

After solving the differential equations, the solution is interpreted by demonstrating its physical meaning. This is accomplished by plotting the first and second derivatives of displacement in time domain. It is then emphasized that the plot of the first derivative of displacement in time domain is a plot of velocity versus time, and the plot of the second derivative of displacement in time domain is a plot of acceleration versus time. The velocity and acceleration plots are not presented in this article due to space considerations.

Summary & Conclusion

In this article, the techniques for modeling one and two degree of freedom vibrating systems are presented. The development of the models involves using calculus and differential

equations. By developing the models the physical interpretation of the mathematical concepts are demonstrated.

Where a readily available solution for the differential equations is available it is presented. The differential equation is then solved by numerical techniques and the results are compared to the exact solutions in order to demonstrate the technique of using computers for numerically solving differential equations.

After developing the basic numerical techniques, they are used in more complex circumstances where closed form solutions are not readily available and the results are presented with the intention of demonstrating the engineering meaning of mathematics and the numerically obtained solutions.

References

1. Theory of vibration with applications, third edition by William T. Thompson, section 2.2.
2. Technical calculus, special edition for the University of Toledo, section 10.7.
3. Schaum's outline series, theory and problems of mechanical vibrations, chapter 2.
4. Numerical Methods in Engineering Practice by Amir Wadi Al-Khafaji & John R. Tooley.

Biographical Information

Dr. Hagigat is an associate professor in the college of engineering of University of Toledo, and he is teaching engineering technology courses. Dr. Hagigat has an extensive industrial background, and he is continuously emphasizing the practical applications of engineering material covered in typical engineering technology courses.